ECE 20875

Python for Data Science

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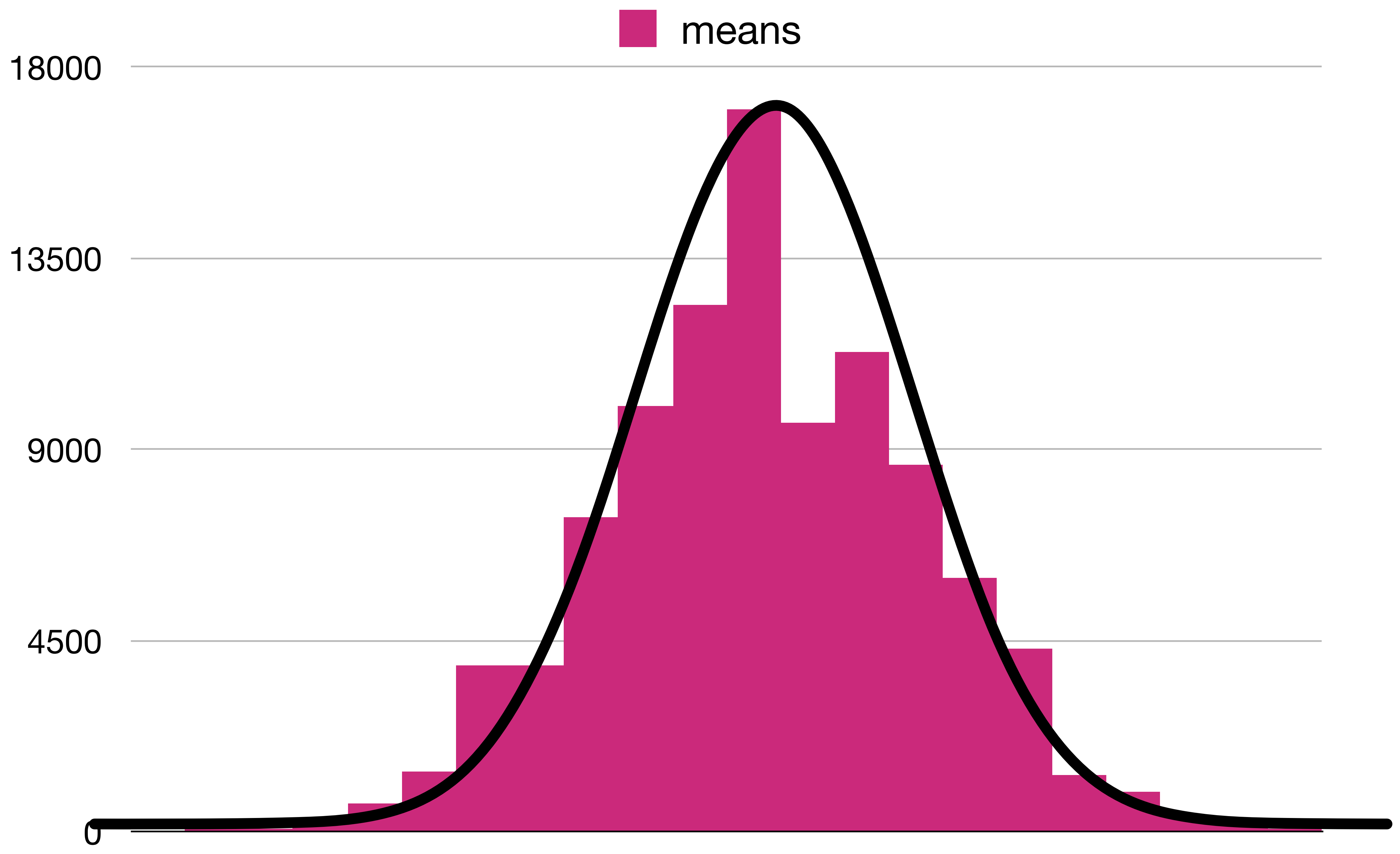
(Adapted from material developed by Profs. Milind Kulkarni, Stanley Chan, Chris Brinton, David Inouye, Qiang Qiu)

confidence intervals and hypothesis testing

sampling distribution

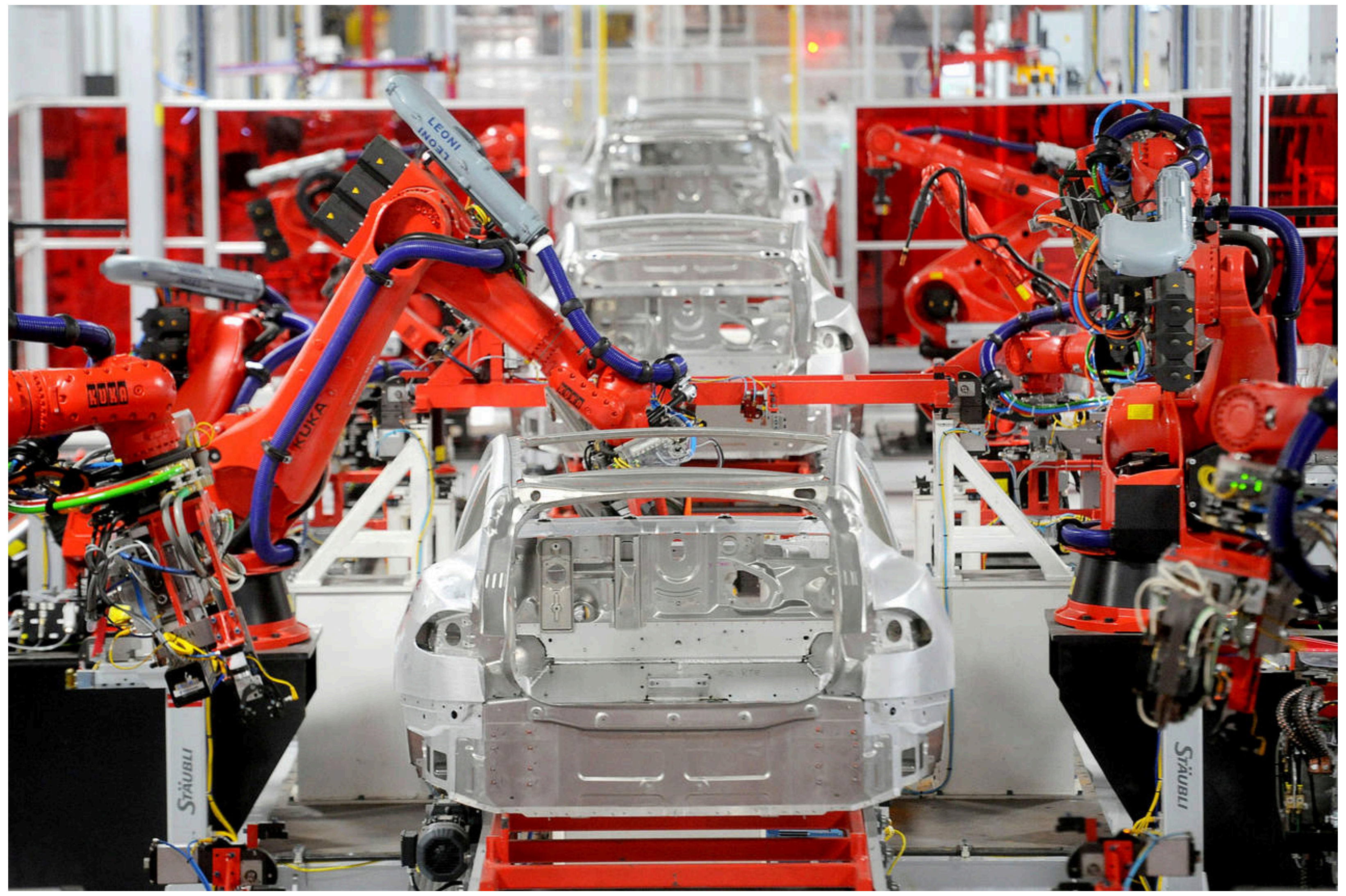


Each data point is the $\bar{x}$ of one experiment

- Recall that by the central limit theorem, sample means approach a normal distribution
- Can we use this to draw conclusions about our data?

asking questions about data

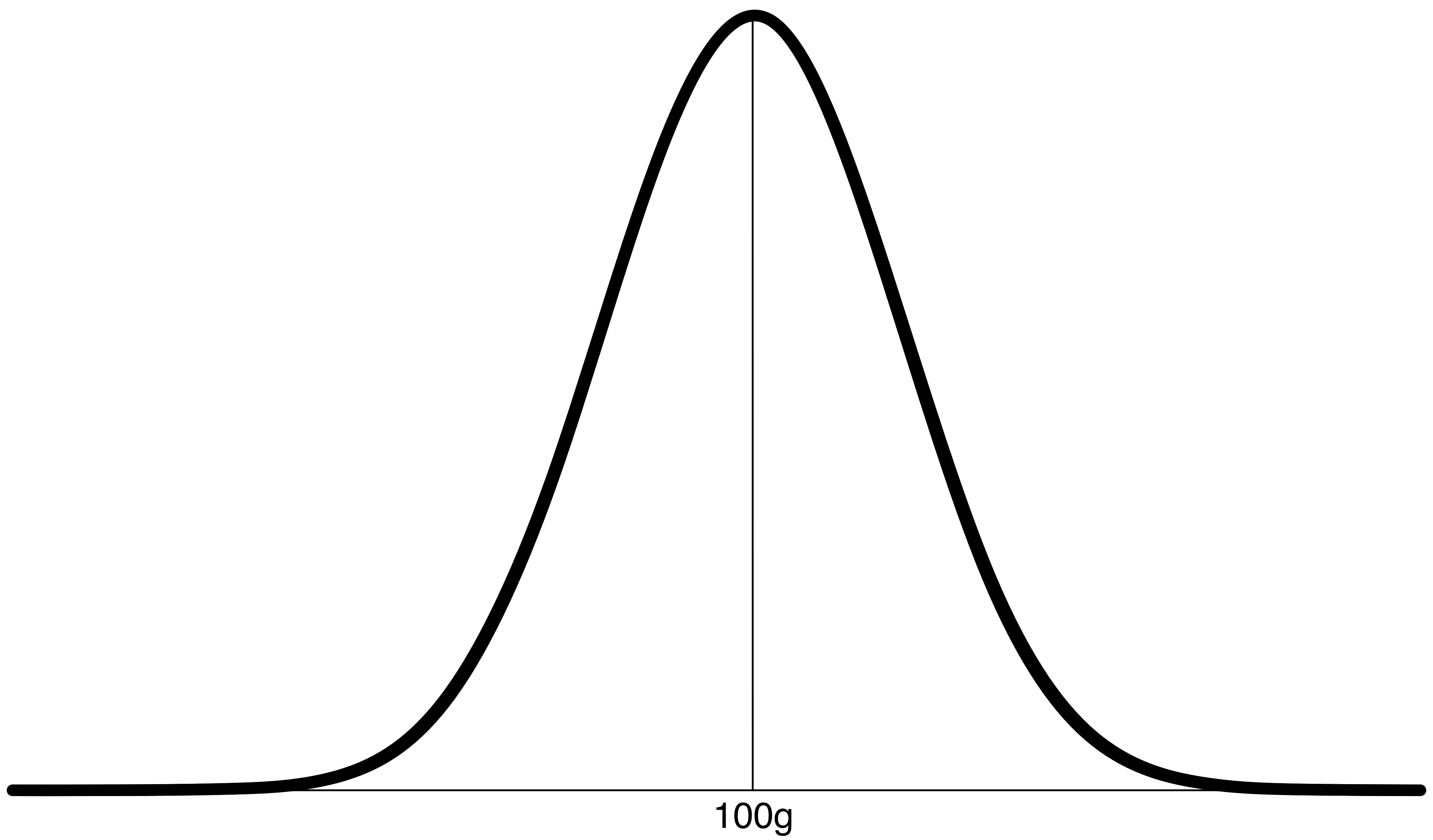
- Suppose a factory claims to produce widgets with an average weight of 100g and a standard deviation of 22g
- We receive a new shipment of widgets which seem off, and we want to see whether the factory has shifted
- Form two hypotheses:
  - **Null hypothesis** (H_0): The factory is producing according to specification, i.e., $\mu = 100g$.
  - **Alternative hypothesis** (H_1): The factory is not producing according to specification, i.e., $\mu \neq 100g$.
- Suppose we weigh 100 of the new widgets (i.e., sample $n = 100$ widgets) and find their average weight is $\bar{x} = 95g$
  - What can we conclude?



asking questions about data

- **Are the widgets in spec?**
- Not as simple as it seems!
- We have picked one sample of widgets, but it could just be a bad sample!
- Can we use our sampling distribution to help?

hypothesis testing



$\mathcal{N}(100, 2.2)$

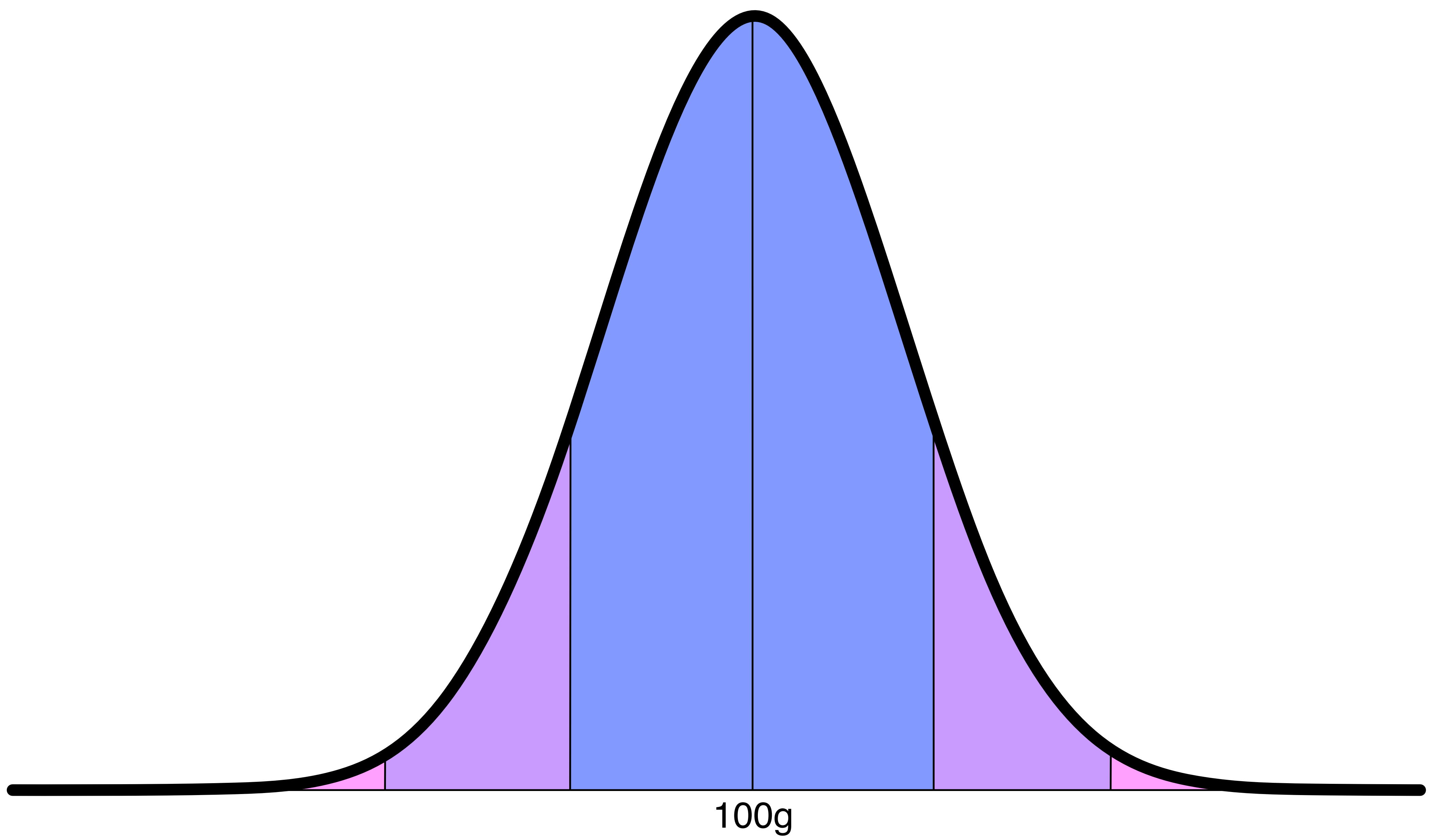
- Suppose the null hypothesis is true (new widgets are from the same distribution as the original widgets)
- Then the sampling distribution should have its mean at $\mu = 100\text{g}$
- And the sampling distribution should have a standard deviation of:

$$SE \triangleq \sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} \approx \frac{22}{10} = 2.2\text{g}$$

  - This is called the **standard error** (SE)
  - Remember, σ is from the population, which we sometimes have to estimate with s (from the sample)

hypothesis testing

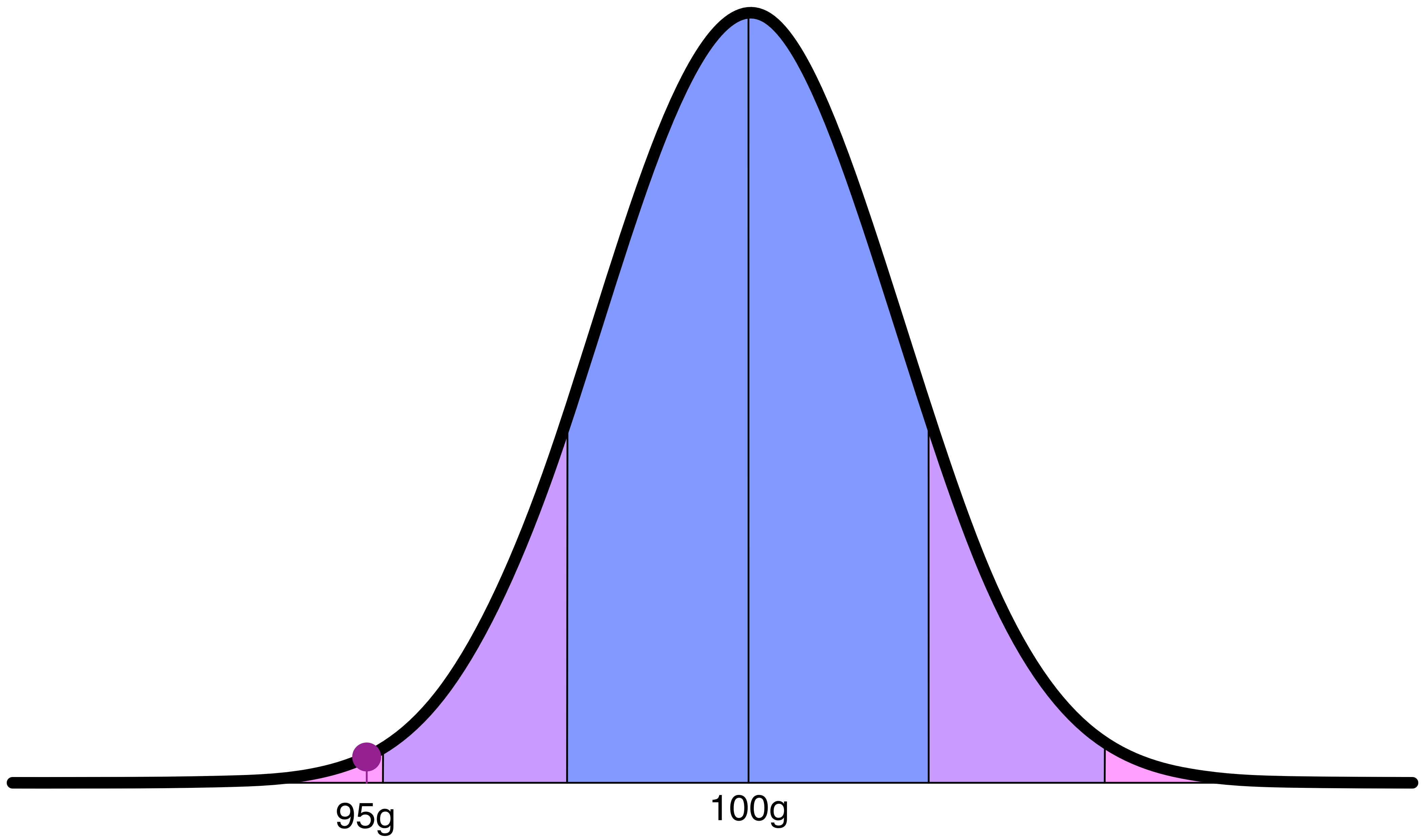
$\mathcal{N}(100, 2.2)$



- Remember properties of normal distribution:
  - ~68% of points within one σ of μ
  - ~95% of points within two σ of μ
  - ~99.7% of points within three σ of μ

hypothesis testing

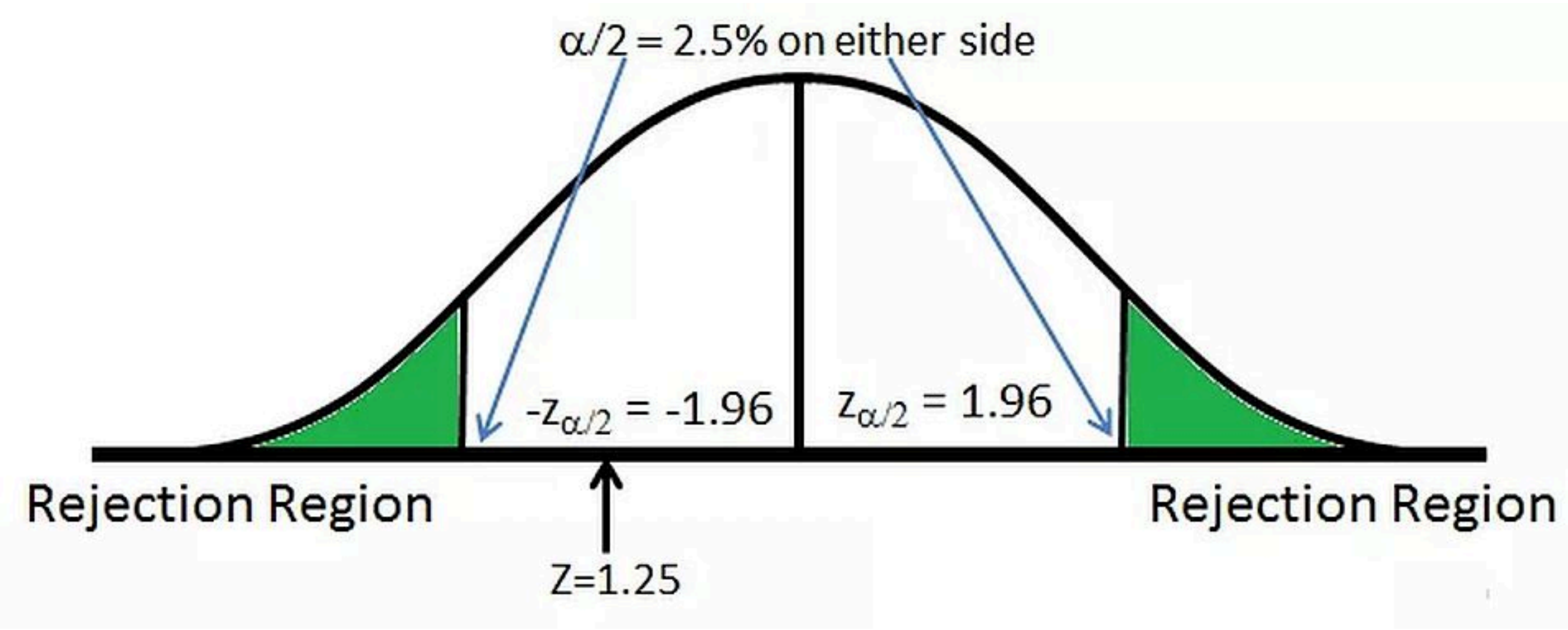
- So what about our sample $\bar{x}$ of 95g?



- Very unlikely for it to have come from this distribution!

- Remember properties of normal distribution:
    - ~68% of points within one σ of μ
    - ~95% of points within two σ of μ
    - ~99.7% of points within three σ of μ
- 95g is between 2 and 3 $\sigma_{\bar{X}}$ of μ

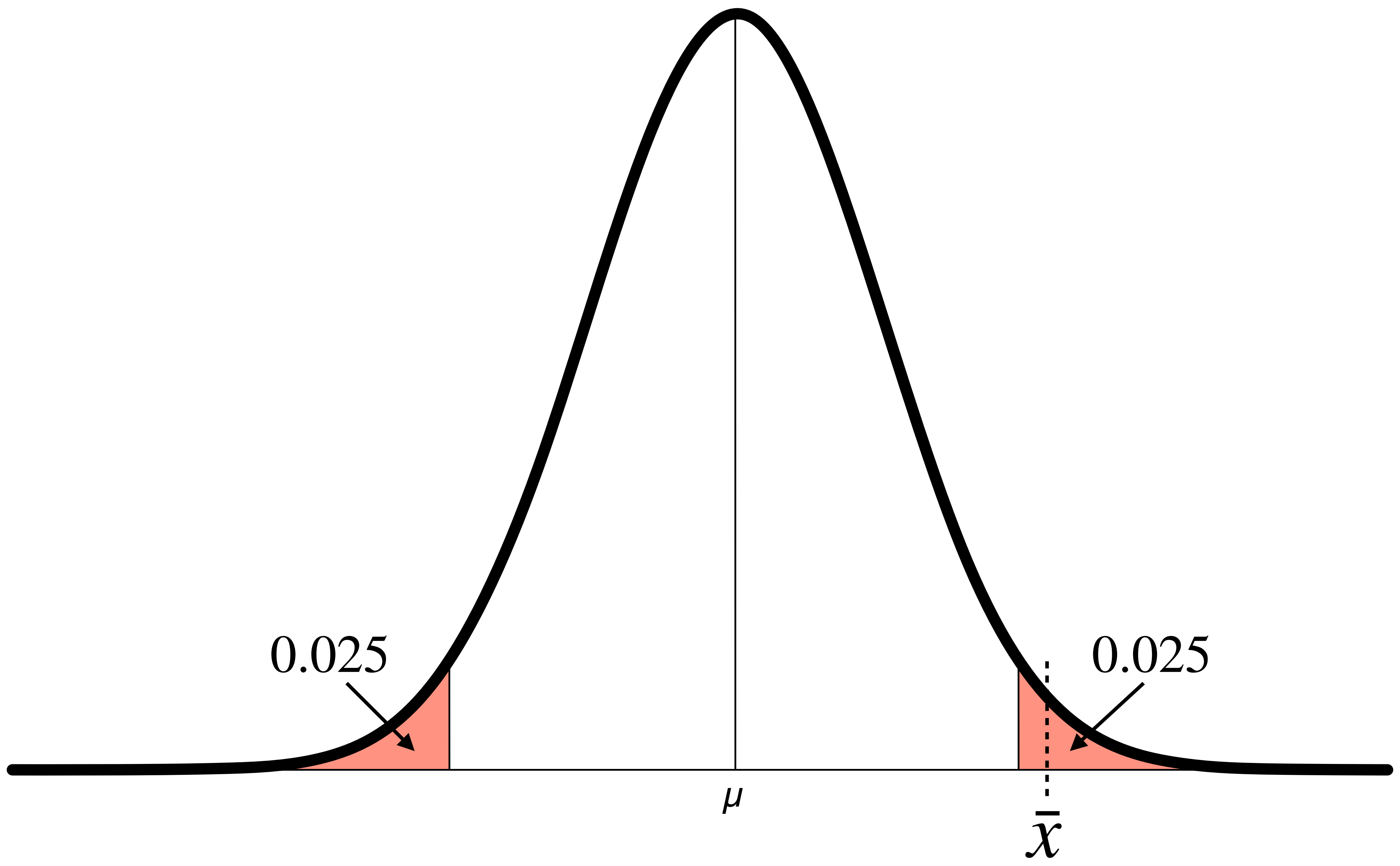
z-test



- The statistical **z-test**
  - Reasoning about μ
  - Applicable when we know σ or if n is large enough (if we don't know σ and n is large enough, we can estimate with *s*)
  - Can construct sampling distribution assuming null hypothesis is true
- Set a **significance level** α for the test
  - Fraction of distribution in each "tail" considered anomalous is $\alpha/2$ (if **two-sided test**)
  - See whether sample $\bar{x}$ falls in that tail
  - If so, *reject* null hypothesis H_0 in favor of alternative H_1; otherwise, *do not reject* (but this does not prove that H_0 is true)

z-test

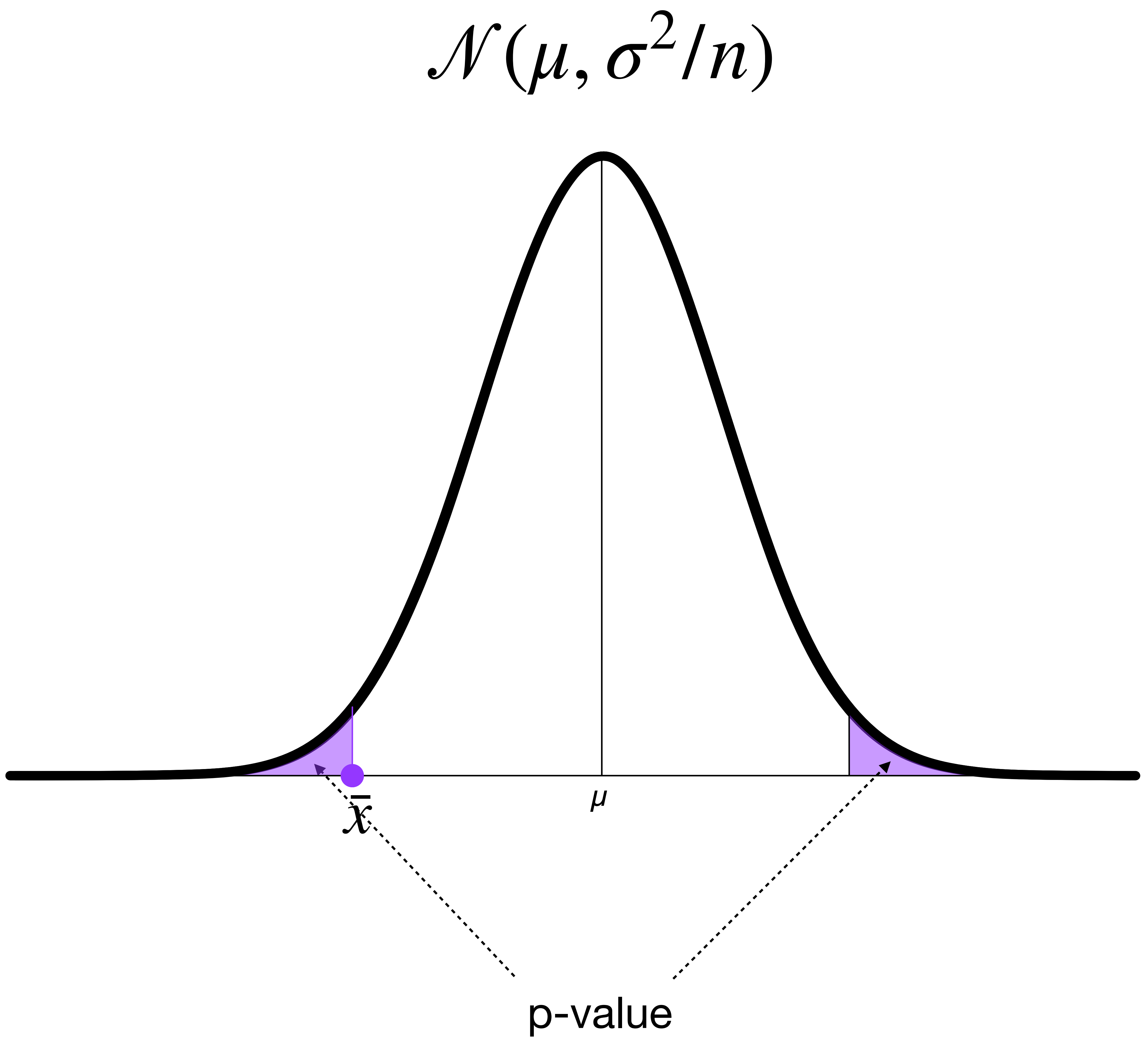
Significance level: $\alpha = 0.05$



In this case, reject H_0

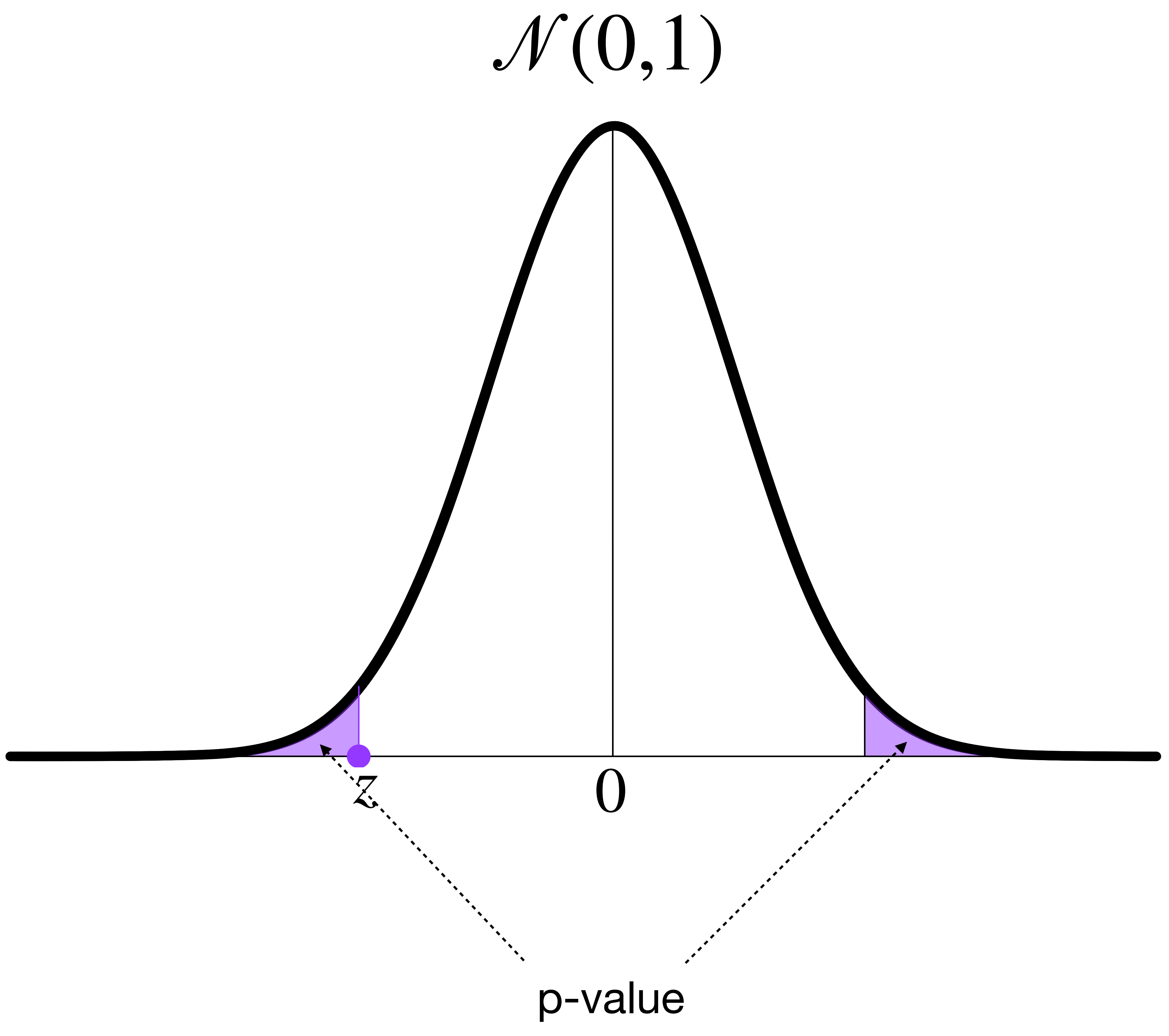
- Set a **significance level** α for the test
  - Fraction of distribution considered anomalous is $\alpha/2$ in each "tail" (if two-sided)
  - See whether sample $\bar{x}$ falls in that tail
  - If so, *reject* null hypothesis H_0 in favor of alternative H_1; otherwise, *do not reject* (but this does not prove that H_0 is true)

p-value for z-test



- We can formalize this logic by calculating the **p-value**
- Place sample $\bar{x}$ on distribution
- Ask what fraction of distribution is farther from the mean μ than the sample $\bar{x}$
- This is your p-value, which is compared to the significance level α:
  - Usually ask for $\alpha = 0.05$ or 0.01 (i.e., so that $p \leq 0.05, 0.01$ for significance)
  - Sometimes $\alpha = 0.1$ is OK

procedure



- Compute sample mean $\bar{x}$
- Compute standard deviation of sampling distribution (standard error)

$$SE = \frac{\sigma}{\sqrt{n}}$$

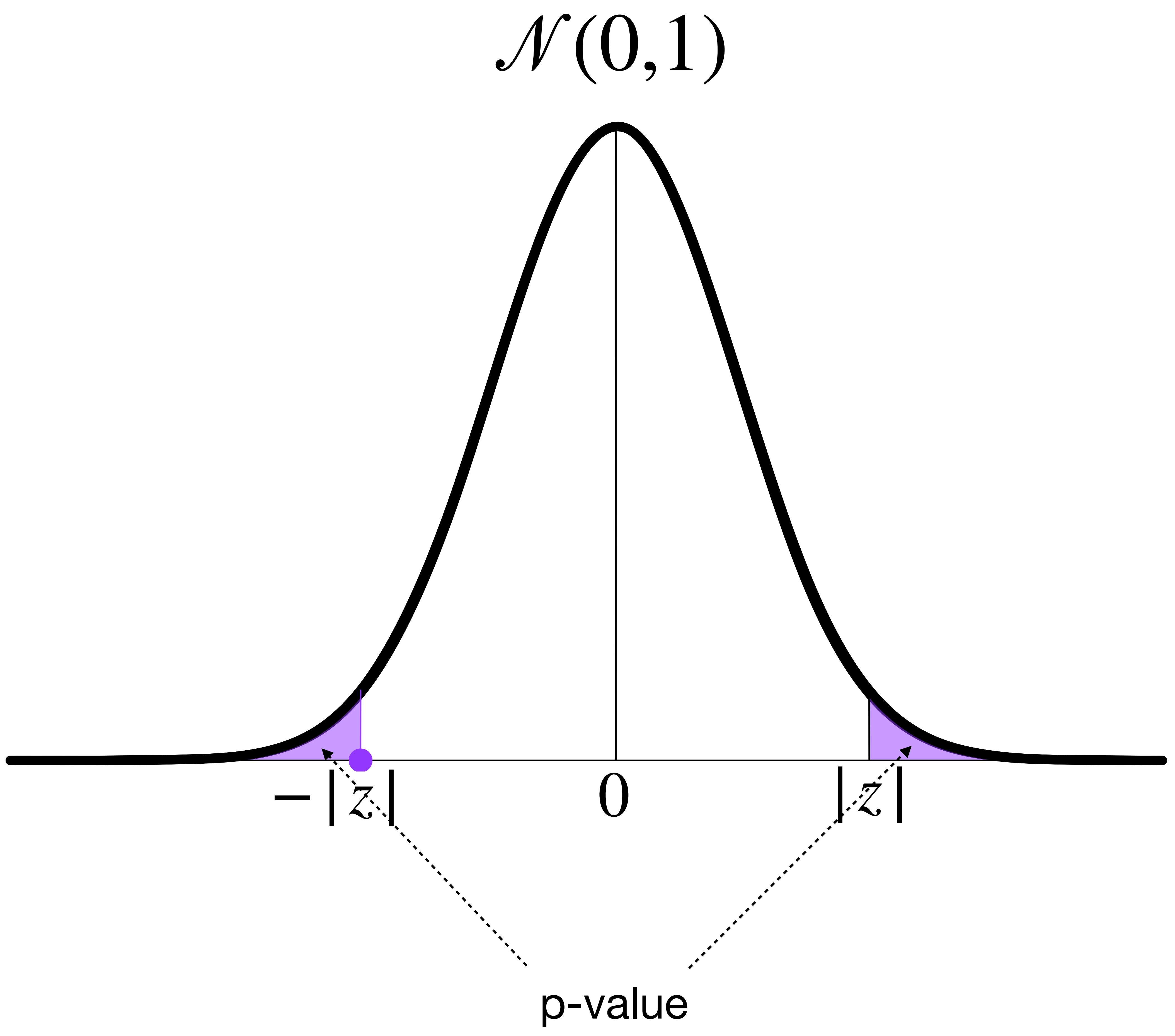
- Compute **z-score**

$$z = \frac{\bar{x} - \mu}{SE}$$

  - Normalizing the sample to the **standard normal distribution** $\mathcal{N}(0,1)$

- Compute p-value from z-score

computing p-value from z-score



- One way: look up in a standard table
- In Python:

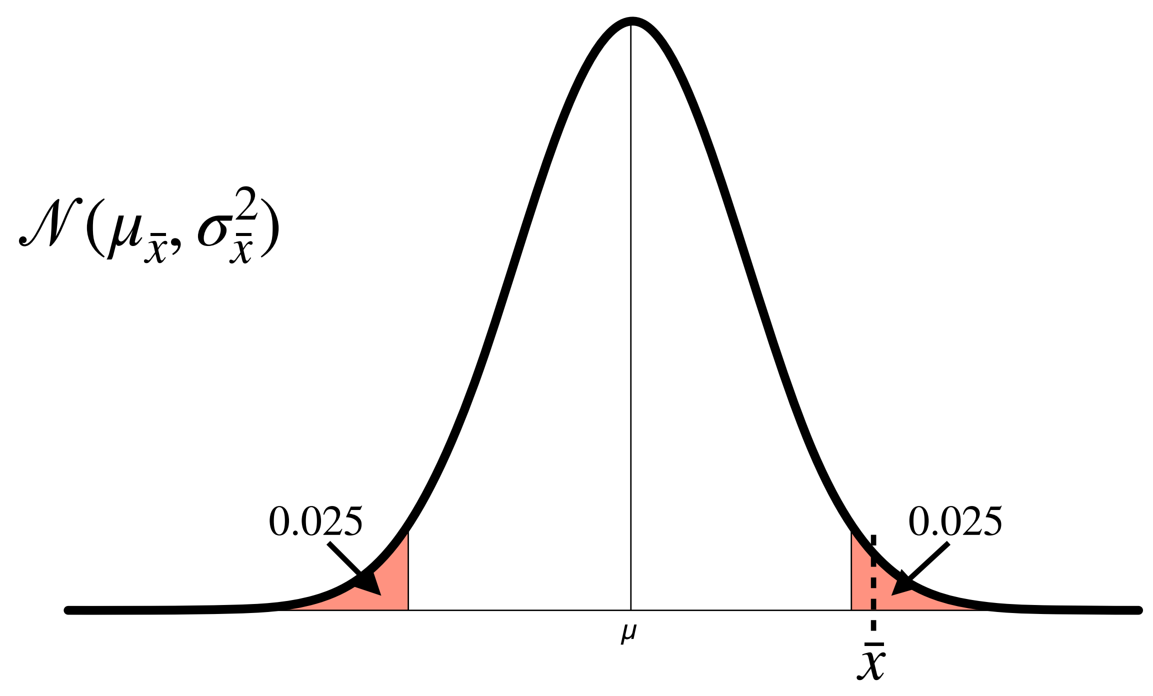
```
import scipy.stats as stats

# compute z = (x − mu) / SE

p = 2 * stats.norm.cdf(−abs(z))
```

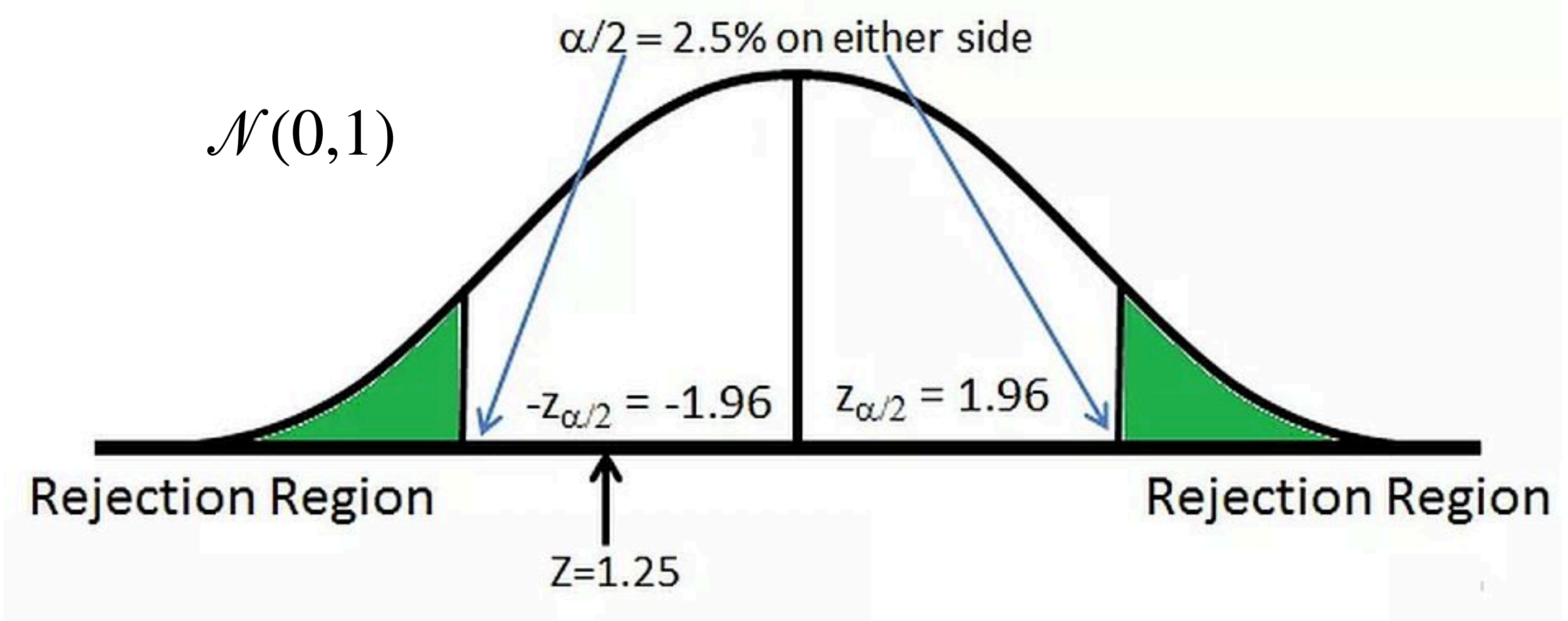
- Why -abs(z)? cdf considers left of the z point, so if z is positive, we want to reference -z

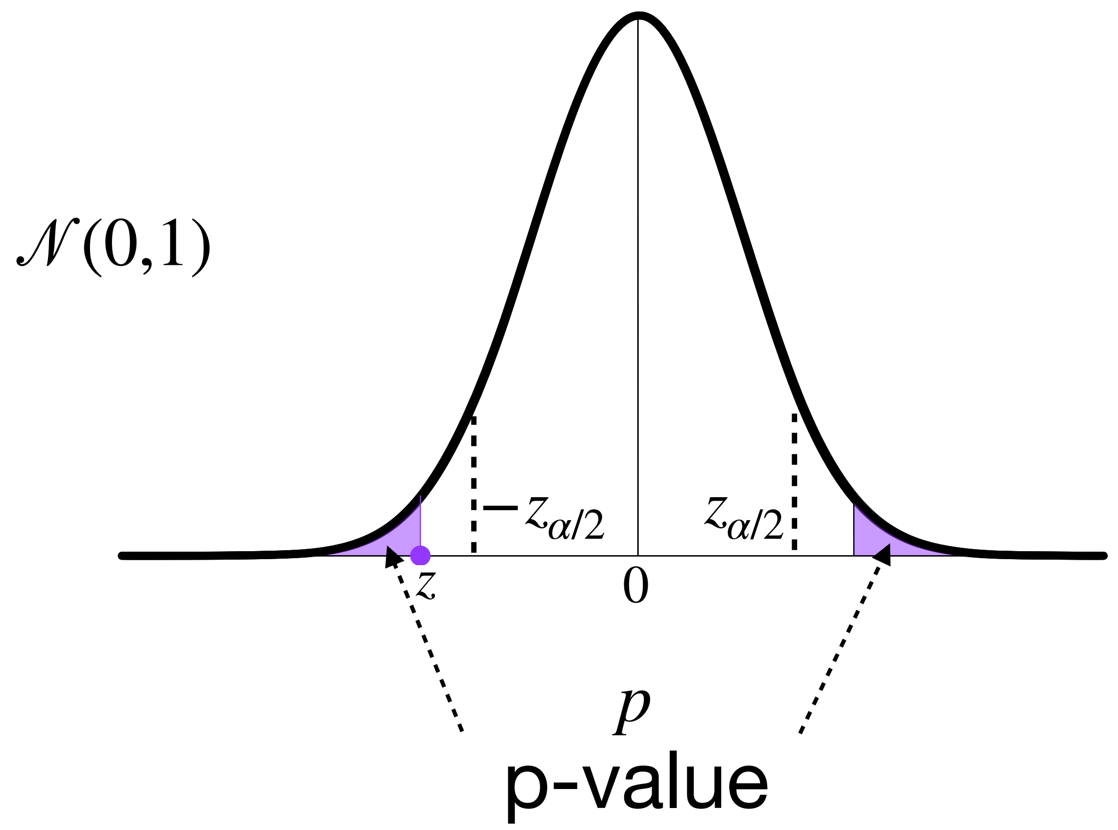
overview of z-test

- Assumptions needed for statistical test
  - Null hypothesis H_0
  - Alternative hypothesis H_1
  - A statistical significance level α
- Equivalent questions (if yes, then reject null hypothesis)
  - Is the sample mean, $\bar{x}$, in tail defined by α of the **sampling distribution** $\approx \mathcal{N}(\mu_{\bar{x}}, \sigma_{\bar{x}}^2)$?
  - Is the z-score, $z = \dfrac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \dfrac{\bar{x} - \mu}{SE}$, in the tail defined by α of a **standard normal** $\mathcal{N}(0,1)$?
  - Is the **p-value**, $p = 2F_{\mathcal{N}(0,1)}(-|z|)$, less than α?

Significance level: $\alpha = 0.05$







back to our original example

- $\mu = 100$, $\sigma = 22$

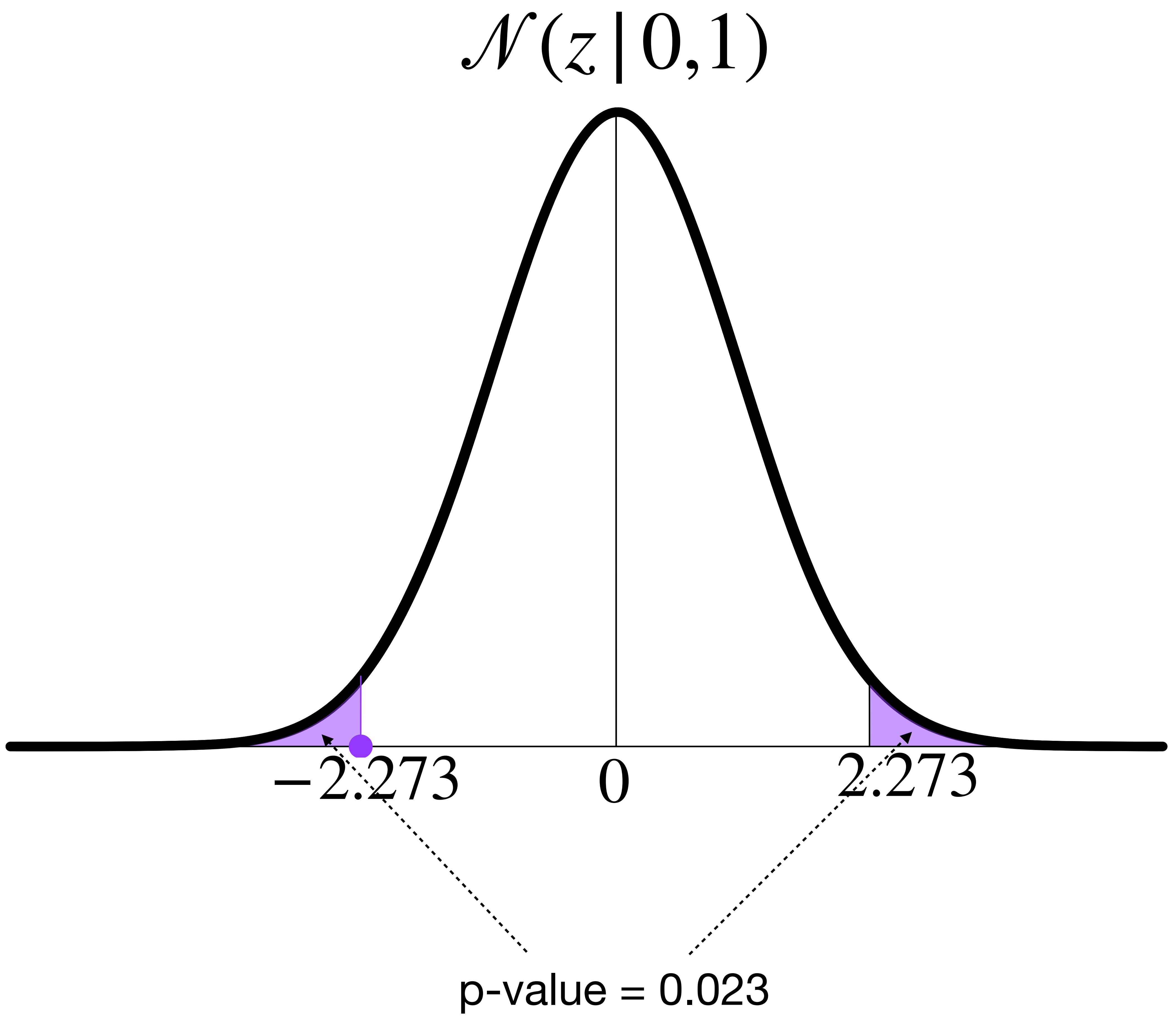
  $\bar{x} = 95$, $n = 100$

- So we calculate:

  $$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{95 - 100}{22/\sqrt{100}} = -2.273$$

  $$p = 2 \cdot F(z \mid 0,1) = 0.023$$

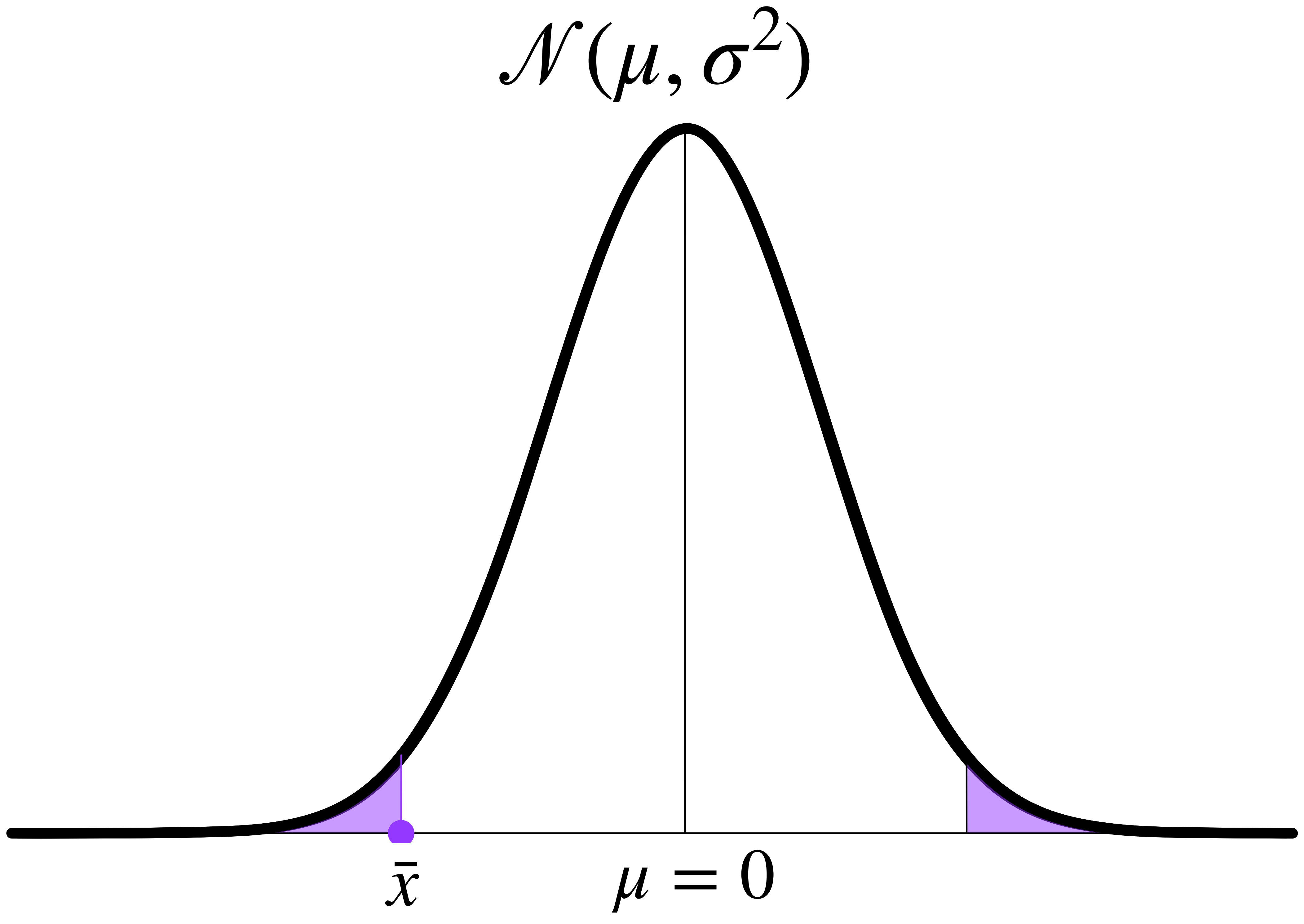
- Conclusion:
  - Significant at $\alpha = 0.1, 0.05$ (reject H_0)
  - Not significant at $\alpha = 0.01$ (cannot reject H_0)



comparing two means

- What if you have *two* populations, and you want to know whether their means are statistically different?
  - Sample 1: Sample size n_0, from pop. mean μ_0, variance σ_0^2
  - Sample 2: Sample size n_1, from pop. mean μ_1, variance σ_1^2
- Hypotheses
  - H_0 : The means are the same, i.e., $\mu_0 = \mu_1$
  - H_1 : The means are different, i.e., $\mu_0 \neq \mu_1$
- Can use **two-sample z-test**
- Under null hypothesis, sampling distribution of *difference between two means* has:

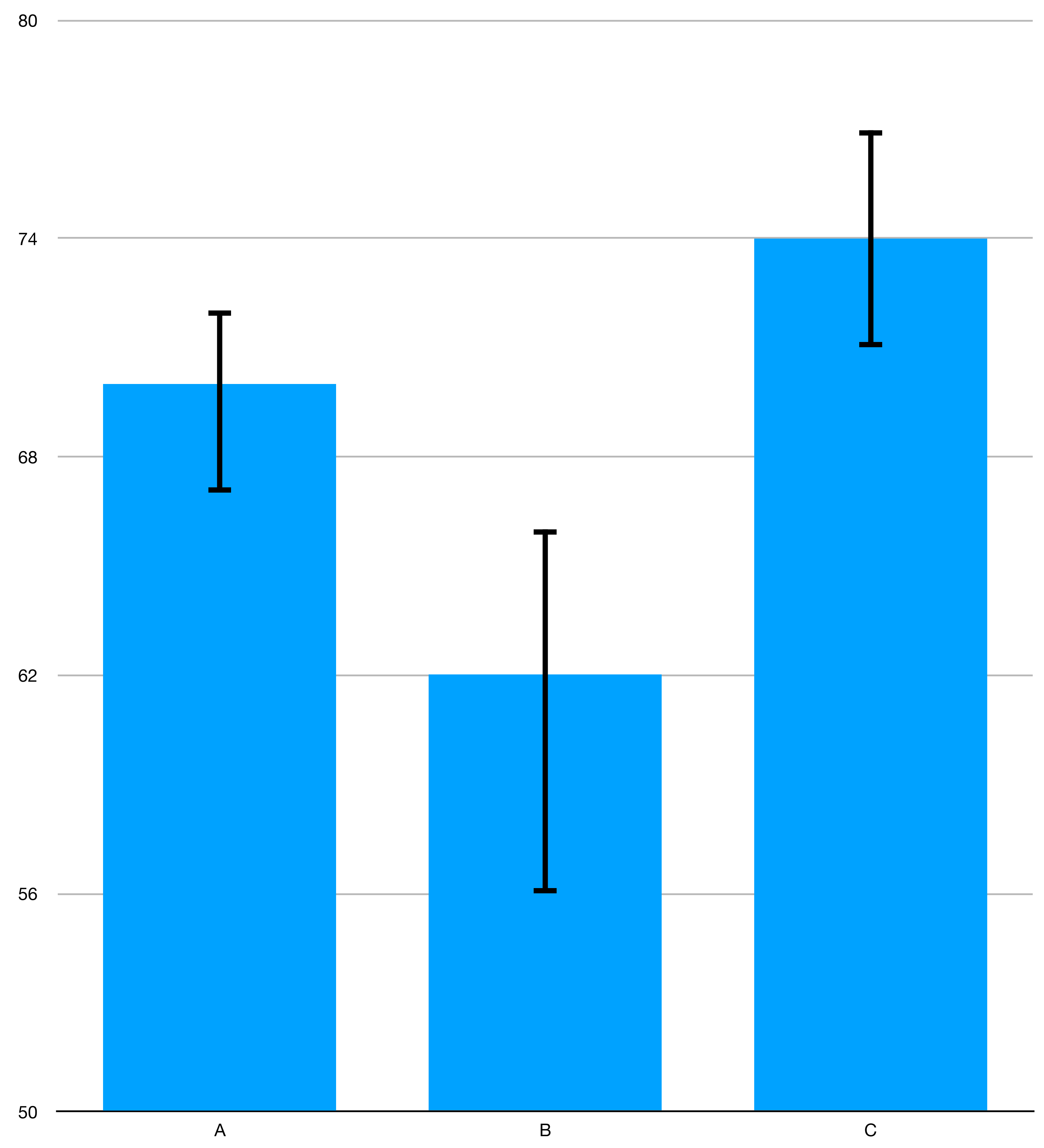
$$\mu = \mu_0 - \mu_1 = 0 \qquad \sigma = \sqrt{\frac{\sigma_0^2}{n_0} + \frac{\sigma_1^2}{n_1}}$$



- Test point is $\bar{x} = \bar{x}_0 - \bar{x}_1$
- z-score is $(\bar{x} - \mu)/\sigma$

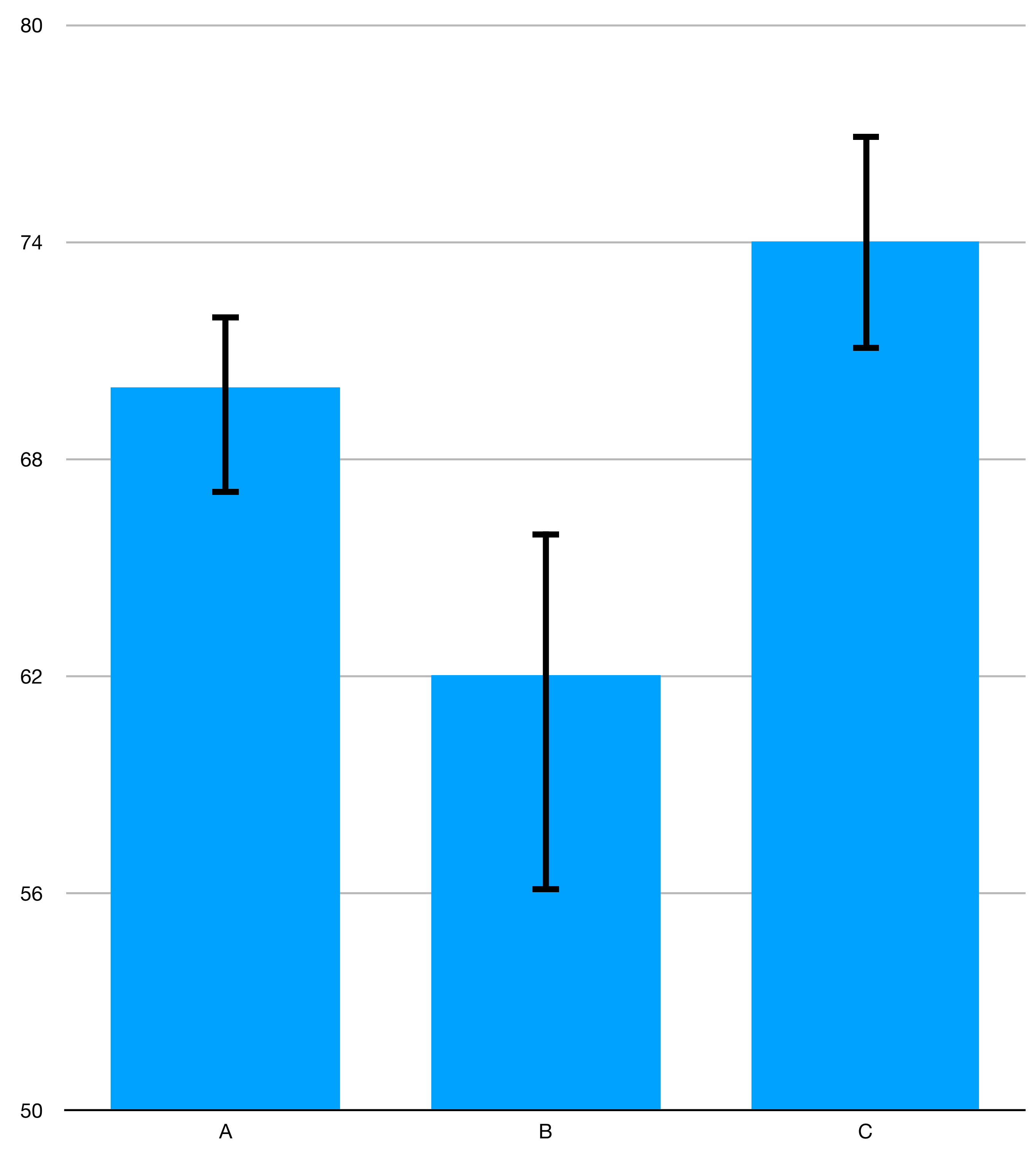
confidence intervals

- We see these a lot: Ranges above and below values on a graph
  - What do they mean?
- Surprisingly tricky question to answer



intuition of confidence intervals

- A **confidence interval** is a range around the mean which says something about how “good” your estimation procedure is
  - How “good” is your choice of number of samples, given the variance in the population
- Interpretation of a (95%) confidence interval:
  - *if I were to repeat the experiment a large number of times, 95 percent of confidence intervals would contain the population mean*
  - *before I run the experiment, there is a 95 percent chance that the population mean will fall within the computed confidence interval*
  - *if the population mean is inside the confidence interval, it would not be statistically significant* (informally, you wouldn’t be surprised!)

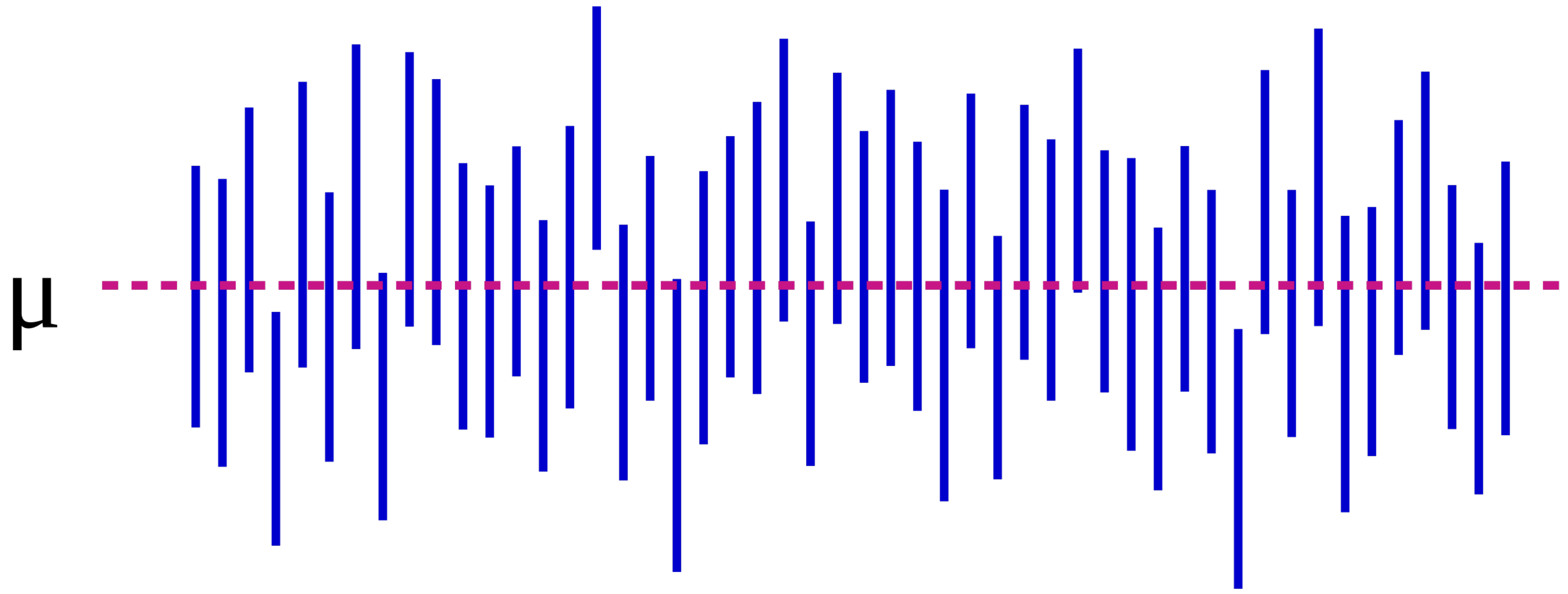


the first interpretation

- *If I were to repeat the experiment a large number of times, 95 percent of confidence intervals would contain the population mean*

- In the diagram below, each vertical bar is one confidence interval calculated for one experiment

    - For a 95% confidence interval, we expect 95% of them will include μ



source: NYW-confidence-interval.svg
Wikipedia user Tsyplakov

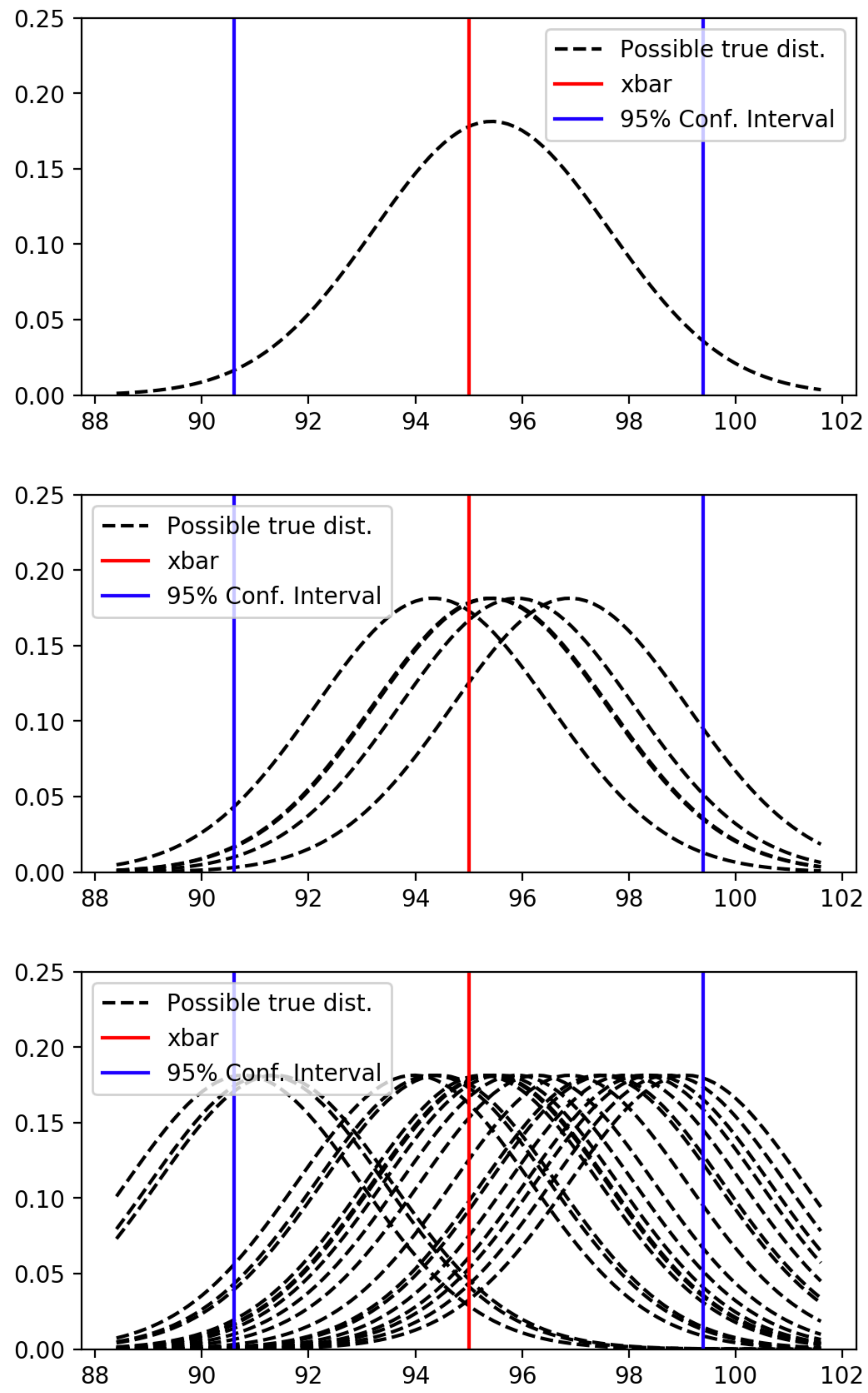
confidence intervals more formally

- If the population parameter is outside the **c% confidence interval**, then an event occurred that had a probability of less than $(100 - c)\,\%$ of happening

- Note that we are setting c ahead of time (unlike with hypothesis testing, where we figure out how likely/unlikely something is *after* the fact)

  - *Wide confidence interval*: The variance of your data is high (and/or your sample size is small), so we need a wide interval to make the above statement true.

  - *Narrow confidence interval*: The variance of your data is small (and/or your sample size is large), so we *don't* need a wide interval to make the above statement true.



computing confidence intervals

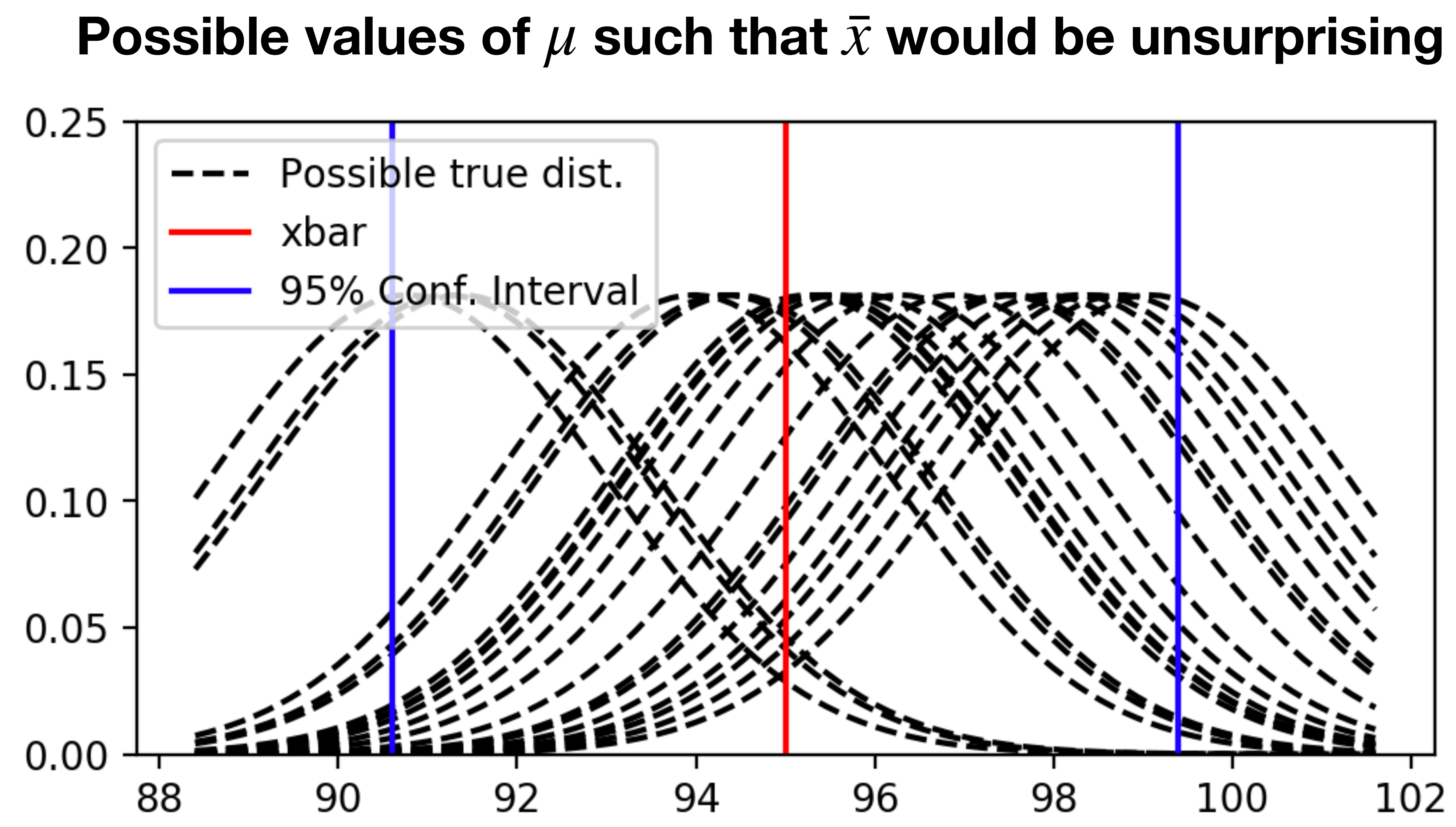
- Conceptually related to z-tests, but the perspective is *reversed*
    - For what sampling distributions (centered at the population mean), would our sample mean NOT be surprising?
    - Note: Our confidence interval is centered around the *sample mean* (instead of the hypothesized population mean)
- Remember definition of z-score:

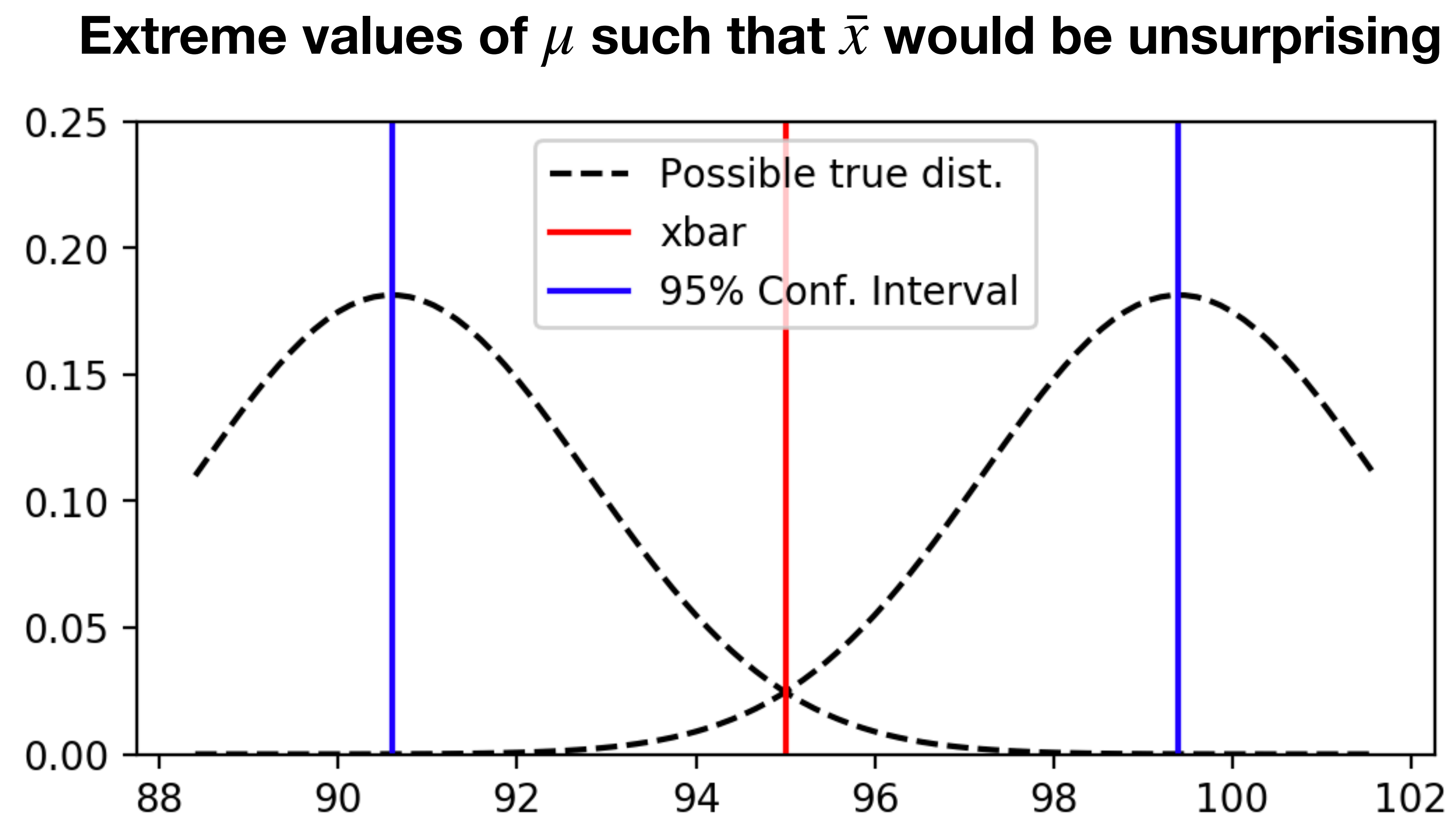
$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

- And p-value:

```
p = 2 * sp.stats.norm.cdf(-abs(z))
```

- If c is the desired confidence level (here in decimal form), what z do we need such that $p \leq (1 - c)$?





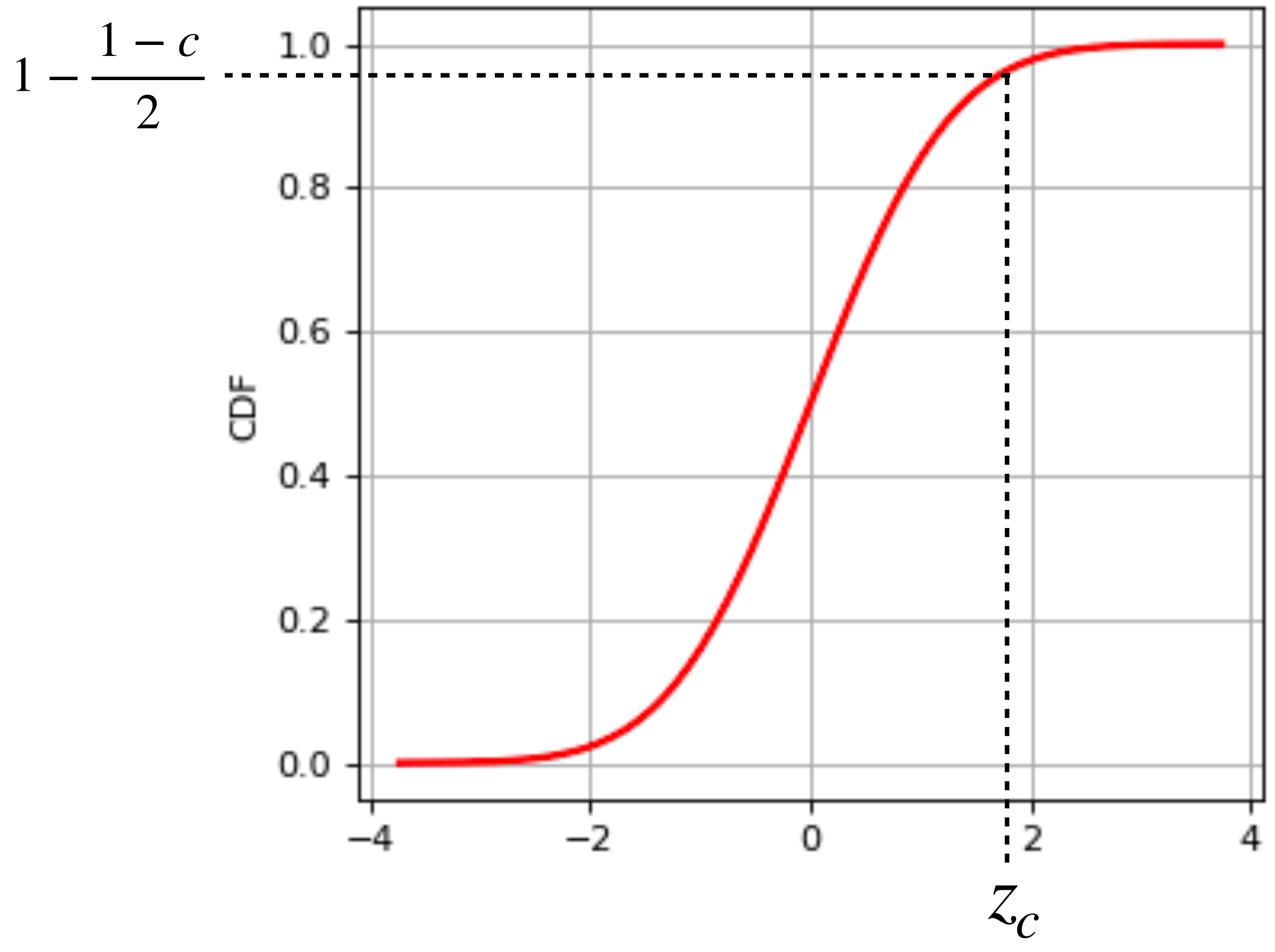
computing confidence intervals

- Call this z_c

- Compute in Python as follows:

```
z_c = stats.norm.ppf(1 - (1 - c)/2)
```

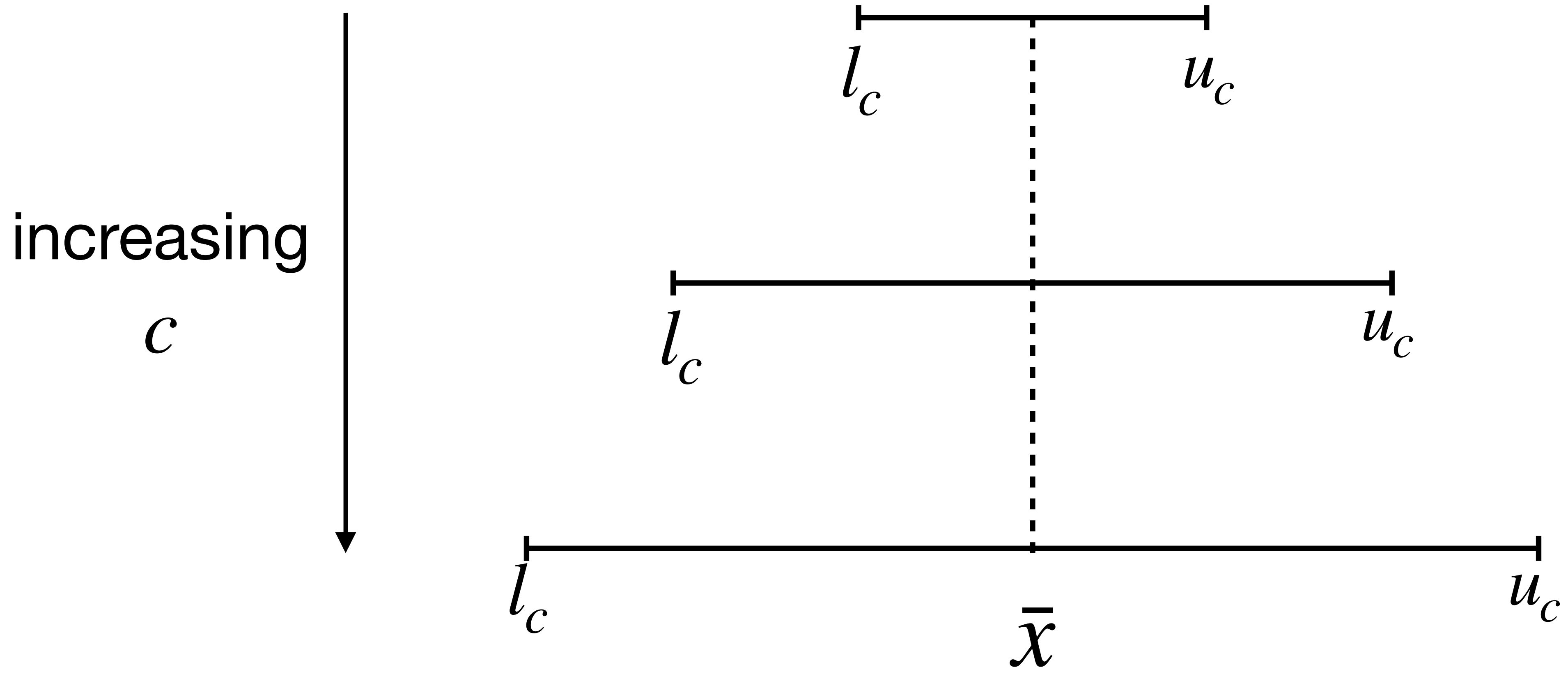
  - While norm.cdf goes from z-score to probability, norm.ppf goes from probability to z-score



- Now we can answer the question: *What range of μ would be "unsurprising" at c% confidence level?*

$$z_c = \left| \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right| \;\rightarrow\; \mu \in \left(\bar{x} - \frac{z_c \cdot \sigma}{\sqrt{n}}, \bar{x} + \frac{z_c \cdot \sigma}{\sqrt{n}} \right) = (l_c, u_c)$$

  - This is your c% confidence interval



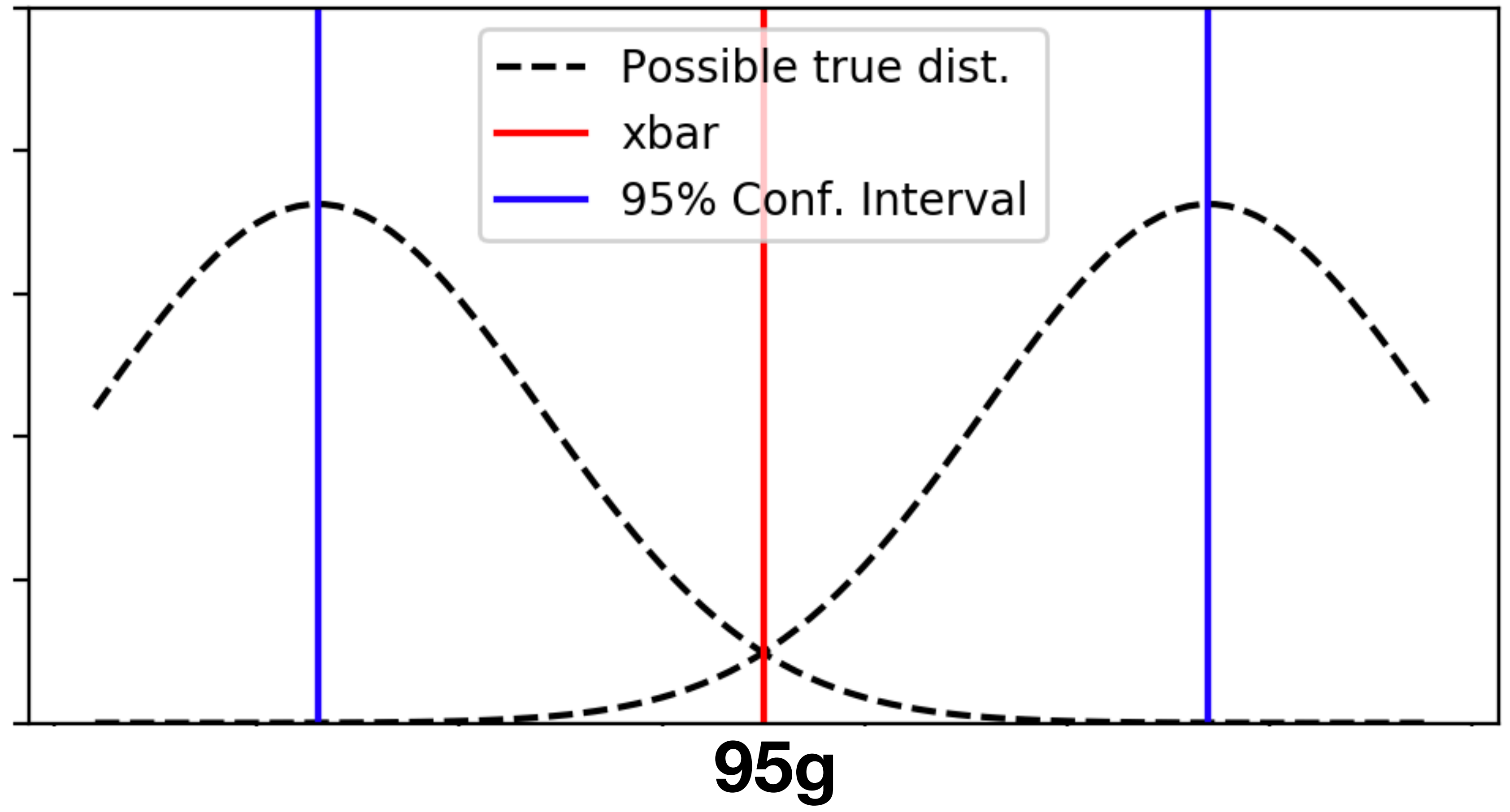
back to our original example

- Let's calculate 90%, 95%, and 99% confidence intervals for μ

- Recall that our sample had

$$\bar{x} = 95g, \sigma = 22g, n = 100$$

- Thus, the confidence intervals are:



$$\mu \in \left(95 - \frac{\sigma}{\sqrt{n}} \cdot z_c, 95 + \frac{\sigma}{\sqrt{n}} \cdot z_c\right) = (95 - 2.2 \cdot z_c, 95 + 2.2 \cdot z_c)$$

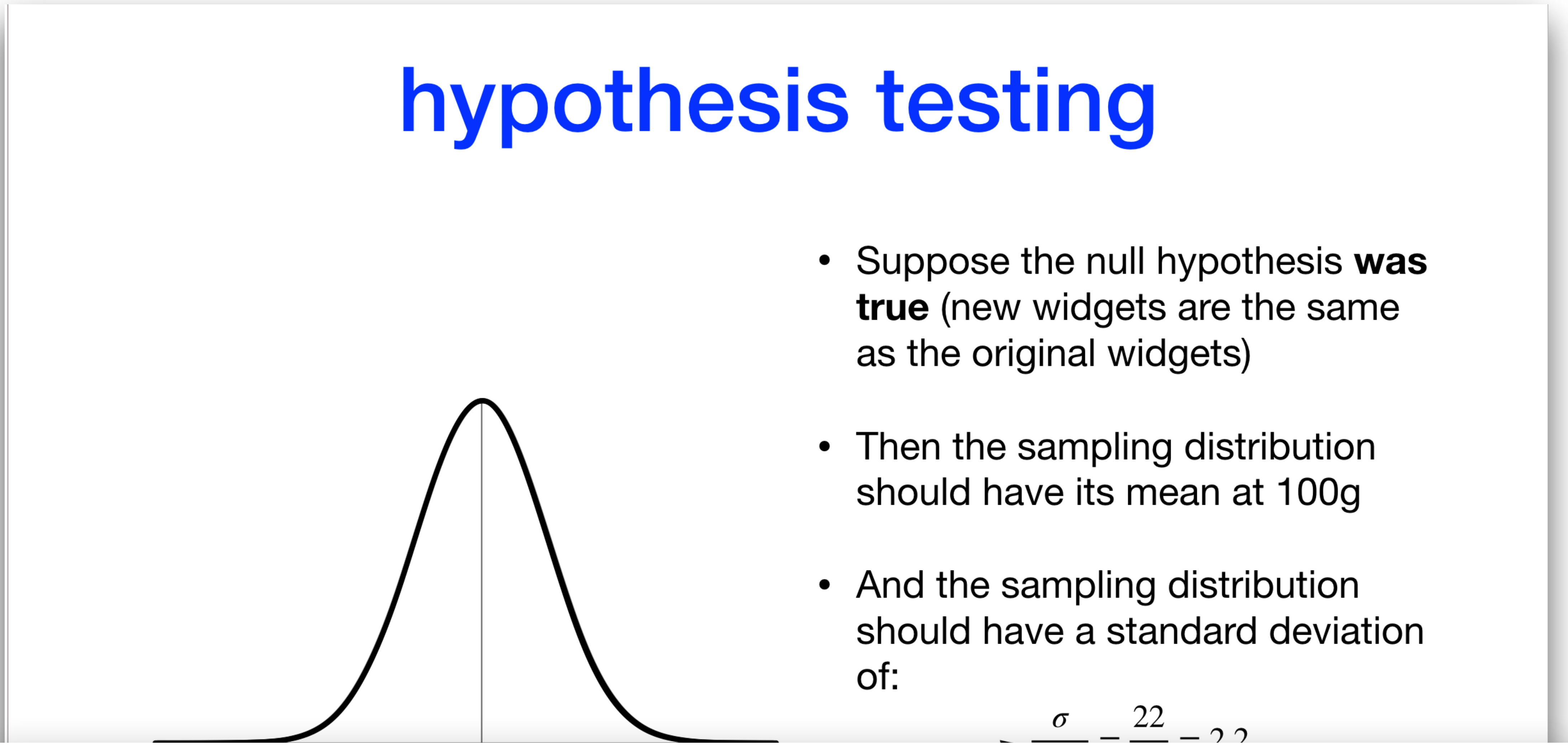
- For 90%, 95%, 99%, $z_c = 1.645,\ 1.960,\ 2.576$. Thus,

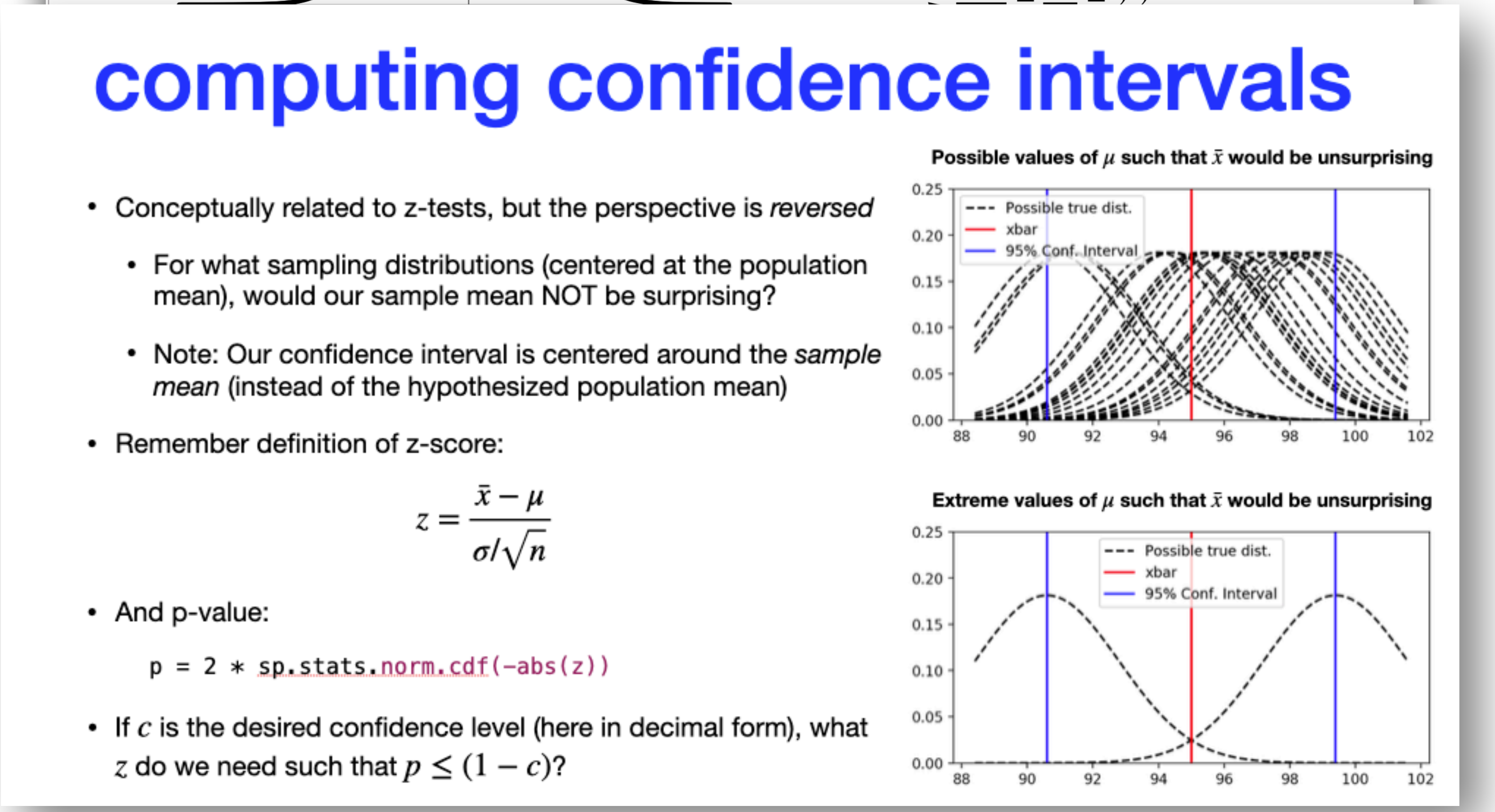
$$90\,\% : (91.38,\ 98.62)$$
$$95\,\% : (90.69,\ 99.31)$$
$$99\,\% : (89.33,\ 100.67)$$

How would we make the intervals narrower for the same levels of confidence?

we’ve been fudging

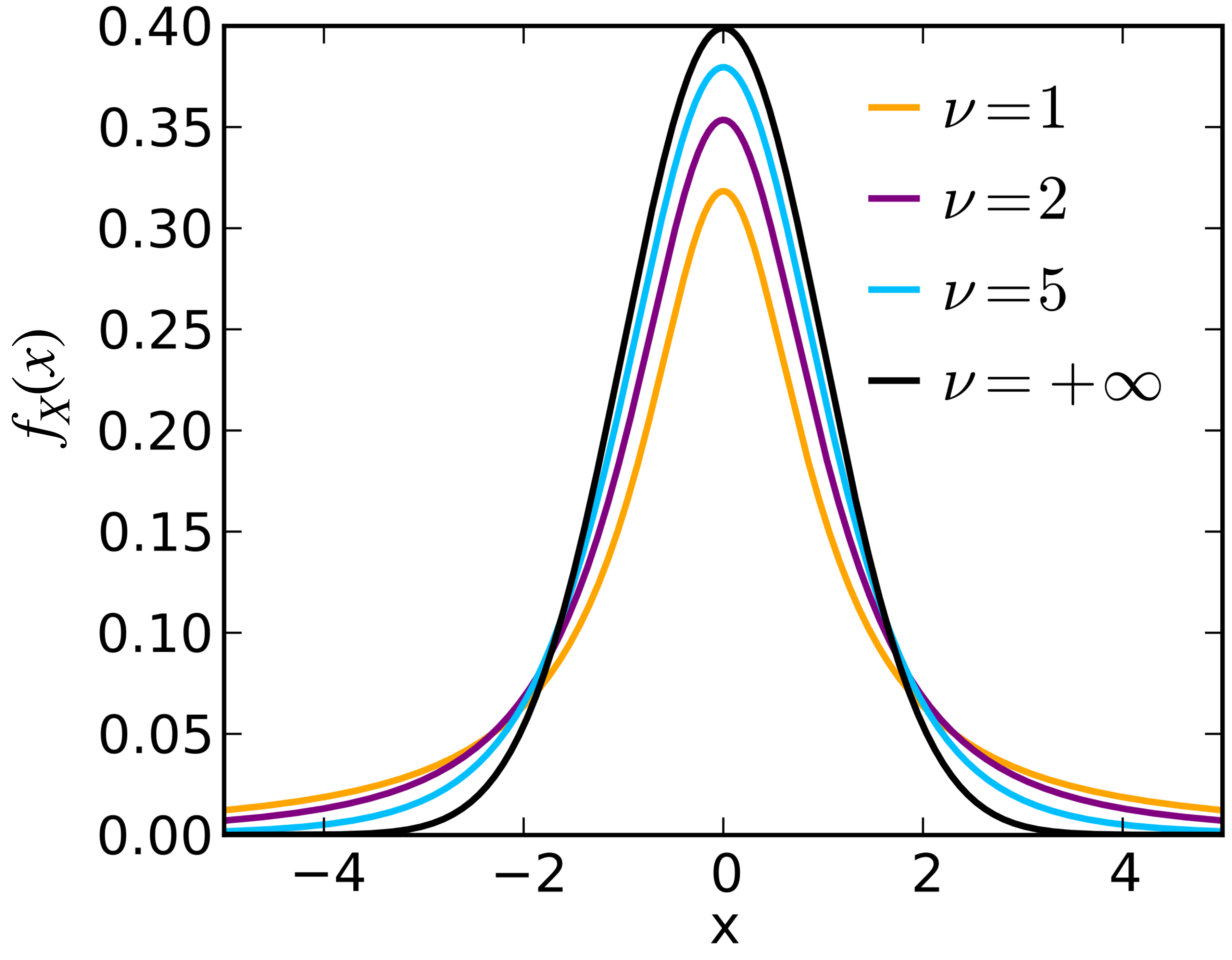
- Recall that to use the z-distribution, we must either know σ or have large enough n
- The **student’s t-distribution** and **t-test** is used when the normal approximation does not hold:
  - i.e., when we don’t know σ (which we usually do not) **<u>and</u>** when $n < 30$
  - Can use this to reason about μ, including building confidence intervals and conducting hypothesis tests





student's t-distribution

- Similar to the standard $\mathcal{N}(0,1)$ normal distribution (density shown to the right)
  - Symmetric about mean
  - Bell curve shaped
- But has **fatter tails**, i.e., more weight of the distribution away from the mean
  - Accounts for outliers better
- Parameter of the distribution is the **degrees of freedom** v
  - $v = n - 1$: One less than the number of samples
  - Looks more and more like the standard normal as $n \to \infty$



t-test and confidence intervals

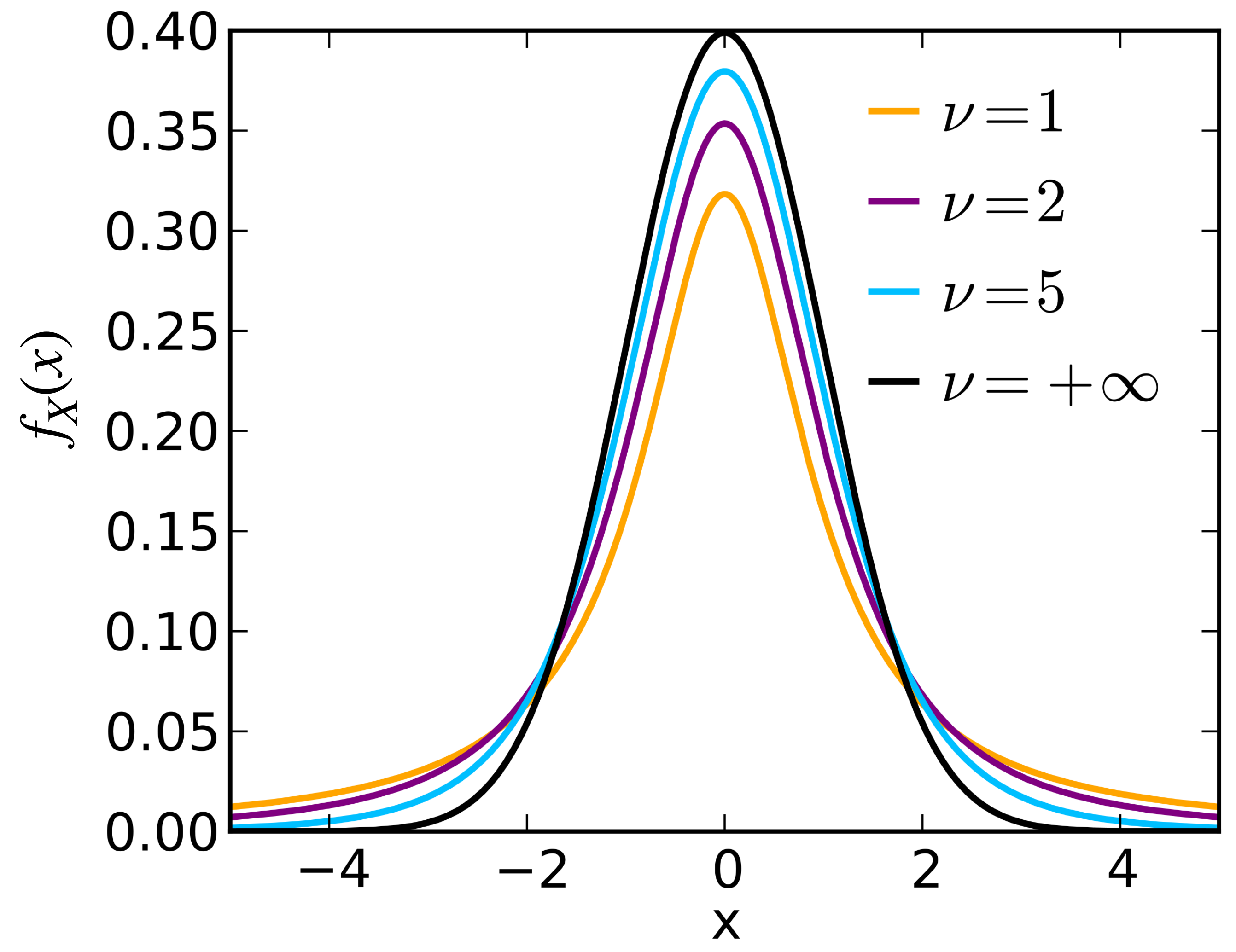
- Works the same as the z-test, except
  - Use s instead of σ
  - Compare to the t-distribution
- Computing the test statistic:
  - First get the standard deviation of the sample:

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

  - Then we get the “t-score”:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

Compare to the formula for z



- Then we get the p-value:

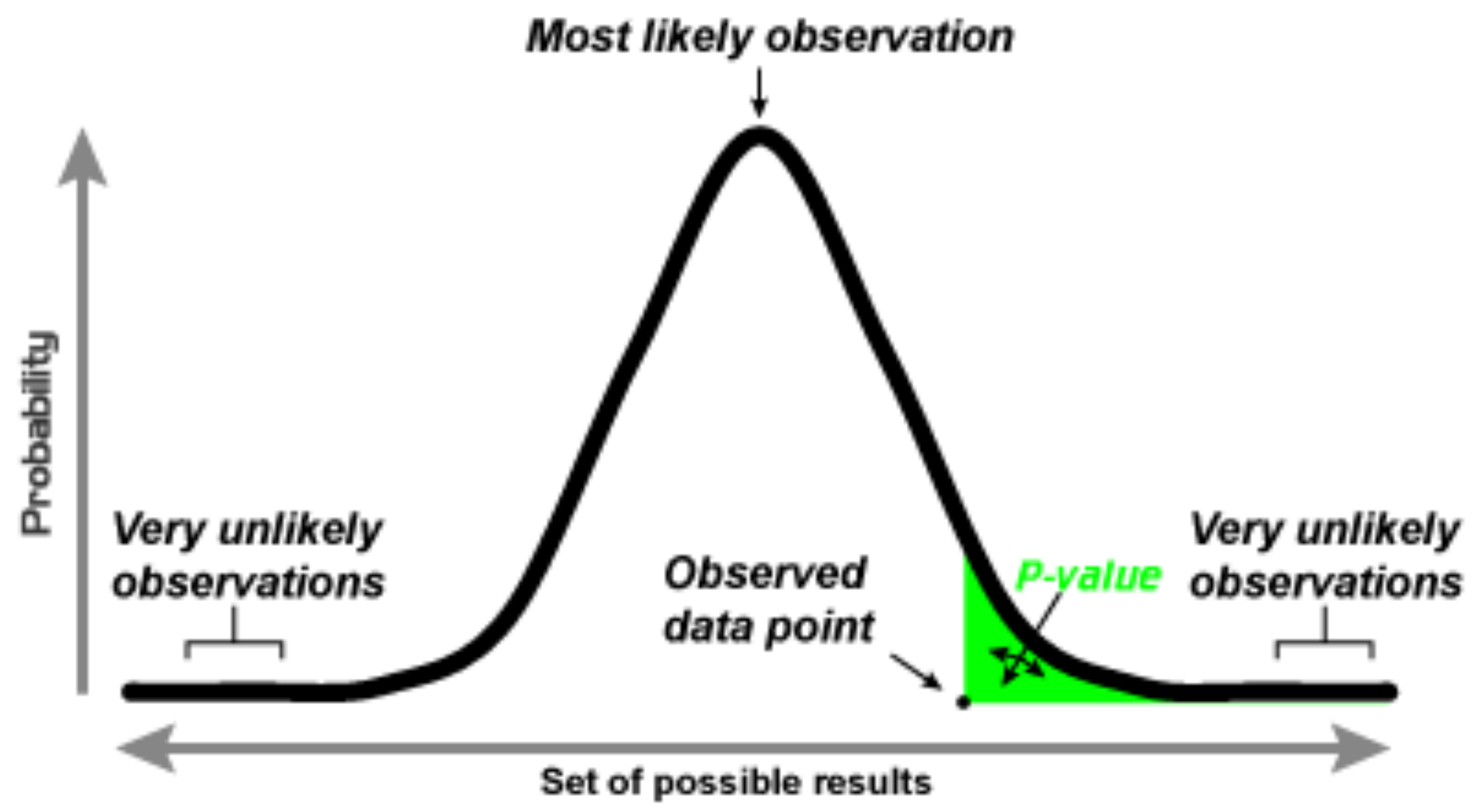
```
p = 2 * stats.t.cdf(-abs(t), df)
```

- And for confidence intervals, we find the t-score corresponding to c:

```
t_c = stats.t.ppf(1 - (1 - c)/2, df)
```

one-sided tests

- Sometimes we are only interested in values departing from the mean in one direction
  - This is a **one-sided** or **one-tailed test**
- For example, suppose we want to assess whether our widgets are being produced at a significantly *higher* weight:
  - $H_0 : \mu \leq 100g$ ← **Null hypothesis is always the logical "opposite"**
  - $H_1 : \mu > 100g$
- How does the p-value compare between one and two-sided tests?



- Any given datapoint has *half* the p-value in a one-sided test than it does in a two-sided test
- We also do not divide α by 2 for a one-sided test, because all the area is now in one tail

simple extensions

- What do we do in a two-sample test when one of the samples violates the normal approximation assumptions?
    - Use a **two-sample t-test**
- Can we build a confidence interval around a mean when the normal approximation is violated?
    - Yes, as discussed, just use the t-statistic in place of the z-score
- What if we are only interested in a confidence interval on one side (e.g., a lower bound or an upper bound)?
    - Can use a **one-sided interval**, where one of the bounds is replaced by $-\infty$ or $+\infty$
    - When computing z_c or t_c, instead of $1-(1-c)/2$ (where dividing by 2), use $1-(1-c)=c$ since there is only one tail