K-Nearest Neighbors
(and Evaluating ML Methods)

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Outline

- K-Nearest Neighbors (KNN) simple algorithm
- Evaluating methods (i.e., generalization error)
  - Train vs test data
  - Cross validation
- Hyperparameter tuning (choosing $k$)
- Curse of dimensionality revisited
The naïve KNN algorithm requires computing the distance to all training points

Input: Test point $x_0$, training data $\{x_i, y_i\}_{i=1}^n$

Output: Predicted class $y_0$

1. Compute distance to all training points:
   \[ d_i = d(x_0, x_i), \forall i \]

2. Sort distances where $\pi$ is a permutation: (e.g., $\pi(1)$ is the index of the closest point)
   \[ d_{\pi(1)} \leq d_{\pi(2)} \leq \cdots \leq d_{\pi(n)} \]

3. Return the most common class of the top $k$
   \[ y_0 = \text{mode} \left \{ y_{\pi(j)} \right \}_{j=1}^k \]
K-nearest neighbors (KNN) is a very simple and intuitive supervised learning algorithm

1. Find the $k$ nearest neighbors
   - Equivalently, expand circle until it includes $k$ points

2. Select most common class

https://www.datacamp.com/community/tutorials/k-nearest-neighbor-classification-scikit-learn
1-1NN partitions the space into Voronoi cells based on the training data

http://scott.fortmann-roe.com/docs/BiasVariance.html
The KNN boundary gets “smoother” as $k$ increases

1-nearest neighbours

20-nearest neighbours

https://kevinzakka.github.io/2016/07/13/k-nearest-neighbor/
How should method performance be estimated?

- Demo on using KNN with training data
How should method performance be estimated? It should be evaluated on unseen test data.

- If we train and evaluate on the same data, the model may not generalize well.

- Analogy to class
  - Training data is like homeworks, sample problems, and sample exams
  - Testing data is like the real exam
We actually care about the method’s performance on **new unseen data**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Data we have</th>
<th>What we want</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medical domain</td>
<td>Disease records for past patients</td>
<td>Predict disease for <strong>new patients</strong></td>
</tr>
<tr>
<td>Photos domain</td>
<td>Human-labeled images</td>
<td>Predict object in <strong>new photos</strong></td>
</tr>
<tr>
<td>Business domain</td>
<td>Historical stock prices</td>
<td>Predict <strong>future stock prices</strong></td>
</tr>
</tbody>
</table>
Estimating **generalization** on unseen data is important for model evaluation and model selection

1. **Model evaluation**
   - Is the model accurate enough to deploy?
   - Example: The business department may decide that the ML predictions will be worthwhile if the accuracy in the real world is above 90% on average.

2. **Model selection**
   - Which of many possible models should be used?
   - Example: Which value of $k$ is best for KNN?
Generalization error measures how much error the model makes on unseen data.

How do we measure generalization error since (by definition) we don’t have new unseen data?

Act like we do! 😊
Generalization error can be estimated by splitting the known dataset

- Split the dataset

  1. The **training dataset** is used to estimate the model

  2. The **test dataset** (or held-out dataset) is used to estimate generalization error.

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Cross-validation (CV) generalizes the simple train/test split to \( M \) disjoint splits (effectively reusing data)

- Repeat the split process \( M \) times
  - Fit new model on train
  - Evaluate model on test
- Note: \( M \) models are fitted throughout process
- Final error estimate is average over all folds

\[
M = 3, M = 5, M = 10 \text{ are common}
\]
Generalization error via CV can aid in model selection (or hyperparameter selection)

(1) Run CV (to estimate generalization) for multiple $k$

(2) Choose $k^*$ whose CV performance is the best

$$k^* = \arg\min_k \text{CVGenError}(k; X)$$

(3) For final model, train model with all data using $k^*$
Back to demo for using cross validation for KNN
But what if we want to select a model AND estimate the model’s performance?

▸ Inception!

▸ Nested train/test split (most common)

▸ Nested CV (better but expensive)
Real-world caveat:
Even CV performance estimates are only good if real-world distribution is like the training data

▶ Training images in the daytime, but real-world images may be at night
  ▶ (Domain generalization tackles this problem)

▶ Training based on historical court cases that are biased against minorities, but real-world court cases should be unbiased
  ▶ (Fairness in AI/ML is a recent popular topic)

▶ Training based on historical stock market data, but real-world stock market has changed
Okay, back to KNN... 😊
KNN regression can be used to predict continuous values

1. Find $k$ nearest neighbors
2. Predict average of $k$ nearest neighbors (possibly weighted by distance)

https://medium.com/analytics-vidhya/k-neighbors-regression-analysis-in-python-61532d56d8e4
The performance and intuitions of KNN degrade significantly in high dimensions (one consequence of the curse of dimensionality).

- The distances between **any two points** in high dimensions is nearly the same.

![Graphs showing distance distributions in different dimensions](https://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote02_kNN.html)

Distance between two random points concentrate around a single value.

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https://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote02_kNN.html
The **curse of dimensionality** is **unintuitive**

**Example: Most space is in the “corners”**

- Ratio between unit hypersphere to unit hypercube
  - 1D: \( \frac{2}{2} = 1 \)
  - 2D: \( \frac{\pi}{\frac{4}{4}} = 0.7854 \)
  - 3D: \( \frac{\frac{\pi}{3}}{8} = 0.5238 \)

- d-dimensions: \( V_d(r) = \frac{n}{\Gamma\left(\frac{n}{2}+1\right)} r^d \)
  - Thus, for 10-D: \( \frac{2.55}{2^{10}} = \frac{2.55}{1024} = 0.00249 \)
Solution 1:
Reduce the dimensionality and then use KNN

Fig. 3. (A) The two-dimensional codes for 500 digits of each class produced by taking the first two principal components of all 60,000 training images. (B) The two-dimensional codes found by a 784-1000-500-250-2 autoencoder. For an alternative visualization, see (B).

Solution 2 (non-KNN): Compute distance to hyperplane instead

Distance to hyperplane is **constant** but pairwise distances between points grows as dimensionality increase.

How do we compute distance to hyperplane?

Dot product with unit normal vector plus constant! \( x^T n + c \)

One view of linear classifiers: 1D projection and then classification

Excellent illustrations from: https://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote02_kNN.html
Related reading and source for KNN curse of dimensionality illustrations

▸ https://www.cs.cornell.edu/courses/cs4780/2018fa/lectures/lecturenote02_kNN.html