Multi-Armed Bandits

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The **multi-armed bandit** problem is inspired by a row of slot machines

- A gambler is the **agent**.
- The row of slot machines is the **environment**.
- The agent can take an **action** by pulling a slot machine's "arm".
- The slot machine payout (or lack thereof) is the **reward** signal.
- **Goal**: Maximize total payout over time.

https://medium.com/growth-book/guide-to-multi-arm-bandits-what-is-it-and-why-you-probably-shouldnt-use-it-ecc9bb2e5a84

Our Simulated Environment

We'll simulate a **Bernoulli Bandit** environment:

- Agent can choose from multiple actions
- Each action gives a binary reward (0 or 1)
- Reward probabilities are drawn from a Beta(1,2) distribution
- Probabilities are **unknown** to the agent
- Agent must estimate probabilities while maximizing reward

Environment Implementation

```python
import torch
import matplotlib.pyplot as plt

class BernoulliBanditEnvironment():
  def __init__(self, n_actions):
    probs = torch.distributions.Beta(1,2).sample((n_actions,))
    self.reward_dists = [
      torch.distributions.Bernoulli(probs=p)
      for p in probs
    ]

  def get_reward(self, action_index):
    return self.reward_dists[action_index].sample((1,))[0]

  def simulate_step(self, agent):
    action_index = agent.select_action()
    reward = self.get_reward(action_index)
    agent.update(action_index, reward)
    return action_index, reward

  def simulate(self, agent, n_steps, verbosity=1):
    cum_reward = 0
    for i in range(n_steps):
      action_index, reward = self.simulate_step(agent)
      cum_reward += reward
      if verbosity >= 1:
        print(
```

Testing the Environment

Let's create an environment and see how it works:

```
1 torch.manual_seed(0)
2 env = BernoulliBanditEnvironment(3)
3 print('Reward distributions')
4 print(env.reward_dists)
5 print('\nRandom rewards for first distribution')
6 print([env.get_reward(0) for i in range(20)])
```

```
Reward distributions
[Bernoulli(probs: 0.6865779757499695), Bernoulli(probs: 0.20974847674369812), Bernoulli(probs: 0.16393446922302246)]

Random rewards for first distribution
[tensor(1.), tensor(1.), tensor(0.), tensor(1.), tensor(1.), tensor(1.), tensor(1.), tensor(0.), tensor(1.), tensor(1.),
tensor(1.), tensor(1.), tensor(1.), tensor(0.), tensor(0.), tensor(0.), tensor(1.), tensor(1.), tensor(1.), tensor(1.)]
```

Notice how the rewards are stochastic (random) but follow the underlying probability.

Agent 1: Interactive Agent (i.e., you!)

```
class InteractiveAgent:
  def __init__(self, n_actions):
    self.n_actions = n_actions

  def select_action(self):
    action_index = -1
    while action_index < 0:
      print(f'Enter action index from 0-{self.n_actions-1}')
      try:
        action_index = int(input())
      except Exception:
        action_index = -1
      if action_index >= 0 and action_index < self.n_actions:
        break
      else:
        print('Incorrect input, try again.')
        action_index = -1
    return torch.tensor(action_index)

  def update(self, action_index, reward):
    return self

torch.manual_seed(2)
env = BernoulliBanditEnvironment(2)
interactive_agent = InteractiveAgent(len(env.reward_dists))
# Uncomment to be interactive
#env.simulate(interactive_agent, n_steps=10)
```

Multi-armed bandits are a simplification of RL yet they retain core RL-specific ideas

- The environment only has a **single state**.
  - "Observing" the environment state is not necessary since it's always the same.
- The environment **does not change** (in the vanilla bandit problem).
  - The **distribution of rewards** does not change over time *or* due to actions.
  - For example, the payout probabilities for each slot machine are fixed.
- At every timestep, the agent can choose **any action**.
- The only problem is **lack of knowledge**.
  - If we knew which machine gave the highest average payout, we would just take that optimal action again and again.
  - If it was supervised learning, only one example of the "correct" action would be enough!
  - The **explore-exploit tradeoff** still exists because of **uncertainty**.

Multi-armed bandits are a simplification of RL yet they retain core RL-specific ideas

- Bandits isolate the unique feature of RL regarding **"feedback"**.
- **Instructive feedback** provides the correct action no matter which action was already taken (e.g., supervised learning).
  - The optimal action a_t^* (equivalently, ground truth label y^*) is the "feedback" given to a supervised learning system **regardless of the actual action** a_t (equivalently, system's prediction $\hat{y}$).
- **Evaluative feedback** provides a reward depending on the action actually taken.
  - The reward signal is a function of the action actually taken a_t, i.e., $R(a_t)$.
  - Thus, the environment **evaluates the actual *action*/decision** made.

How do we design a **policy** that maximizes the sum of rewards?

- We could just do a completely random policy that randomly chooses an action at every time step

$$A_t \sim \text{Uniform}(\{1, 2, \dots, K\})$$

- This is good because it is **simple** and achieves an average reward over all choices.
  - It chooses good and bad actions evenly.
  - It completely ignores the past (i.e., ignores its experience).
- However, it will often take an action that gives **suboptimal reward**.

A better approach is to estimate the value of each action to determine optimal actions

- First, we will define the **value** of an action as the expected reward given this action:

$$q_*(a) \triangleq \mathbb{E}[R_t | A_t = a]$$

  - R_t represents the reward random variable at time t.
  - A_t represents the action random variable at time t.
  - a represents a specific action.
- If we knew the q_*, then the problem would be trivial, just repeatedly take $A_t = a^* = \arg\max_a q_*(a)$
- Obviously, we do not know q_* but we can **approximate it** given our previous actions:

$$Q_t(a) \approx q_*(a)$$

A **sample average** can be used to estimate the expectation

- We can estimate q_* by using a sample average over the past actions and rewards:

$$Q_t(a) \triangleq \frac{\text{sum of rewards when } a \text{ taken prior to } t}{\text{number of times } a \text{ taken prior to } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{I}(A_i = a)}{\sum_{i=1}^{t-1} \mathbb{I}(A_i = a)}$$

- As an example, suppose the past rewards and actions are:
  - $A = [1, 2, 2, 1, 2, 2, 2]$
  - $R = [0, 1, 1, 1, 0, 1, 1]$
- If $t = 6$, then $Q_6(1) = 1/2, Q_6(2) = 2/3$.
- What would it be for $t = 3$?

Given an estimate of the action value $Q_t(a)$, how could we use this information?

- The **greedy action** optimizes the action value approximation $Q_t(a)$

$$A_t = \arg\max_a Q_t(a)$$

- This is good because it approximates the optimal action $a^* = \arg\max_a q_*(a)$
  - Thus, it will tend to have better reward than the random policy.

- Greedy algorithm
  - Initialize $\forall a, Q_1(a) \leftarrow 0, n_a \leftarrow 0$
  - For $t = \{1, 2, \dots, T\}$
    - Choose $A_t \leftarrow \arg\max_a Q_t(a)$
    - Receive reward $R_t \leftarrow \text{Environment}(A_t)$
    - Update $Q_{t+1}(A_t) \leftarrow \frac{Q_t(A_t) \cdot n_{A_t} + R_t}{n_{A_t} + 1}$
    - Update $n_{A_t} \leftarrow n_{A_t} + 1$

Greedy can be suboptimal if Q_t is a bad approximation

- However, greedy can be **bad**, if $Q_t(a)$ is a **bad approximation**

$$\max_a Q_t(a) \neq \max_a q_*(a)$$

- Thus, the core explore-exploit tradeoff remains:
  - **Exploit** – Choose greedy action to maximize rewards.
  - **Explore** – Choose non-greedy action to improve estimate of Q_t.
- Note: The "explore" part is just about improving our understanding about the environment rather than finding new environment states because there is **only one state** in bandits.
- Can we do better than greedy?

ϵ-Greedy algorithm slightly modifies the greedy algorithm to improve exploration

- One simple idea is to randomly sample arms initially and then do greedy from then on
- A more common approach is to randomly choose between explore (via random algorithm) and exploit (via greedy algorithm).
- **ϵ-Greedy algorithm:**
  - Initialize $\forall a, Q_1(a) \leftarrow 0, n_a \leftarrow 0$
  - For $t = \{1, 2, \dots, T\}$
    - With probability ϵ, choose $A_t \leftarrow \text{RandomAction}()$
    - Otherwise, choose $A_t \leftarrow \arg\max_a Q_t(a)$
    - Receive reward $R_t \leftarrow \text{Environment}(A_t)$
    - Update $Q_{t+1}(A_t) \leftarrow \frac{Q_t(A_t) \cdot n_{A_t} + R_t}{n_{A_t} + 1}$
    - Update $n_{A_t} \leftarrow n_{A_t} + 1$

Demo: Oracle Agent (The Cheater!)

Concept: This agent "cheats" because it knows the true probabilities.

- Always selects the optimal action
- Provides an **upper bound** on performance

```python
class OracleAgent:
  def __init__(self, env):
    self.optimal_action = torch.argmax(torch.tensor([
      dist.probs for dist in env.reward_dists
    ]))
  def select_action(self):
    return self.optimal_action
  def update(self, action_index, reward):
    return self
  def __str__(self):
    return 'Oracle'

torch.manual_seed(2) # 2 and 5
env = BernoulliBanditEnvironment(2)
oracle = OracleAgent(env)
env.simulate(oracle, n_steps=10)
```

```
Step=01, Action=1, Reward=0, Cum reward=0, Avg reward=0.00,
Step=02, Action=1, Reward=1, Cum reward=1, Avg reward=0.50,
Step=03, Action=1, Reward=1, Cum reward=2, Avg reward=0.67,
Step=04, Action=1, Reward=0, Cum reward=2, Avg reward=0.50,
Step=05, Action=1, Reward=1, Cum reward=3, Avg reward=0.60,
Step=06, Action=1, Reward=1, Cum reward=4, Avg reward=0.67,
Step=07, Action=1, Reward=1, Cum reward=5, Avg reward=0.71,
Step=08, Action=1, Reward=0, Cum reward=5, Avg reward=0.62,
Step=09, Action=1, Reward=1, Cum reward=6, Avg reward=0.67,
Step=10, Action=1, Reward=1, Cum reward=7, Avg reward=0.70,

tensor(7.)
```

Challenge: Can You Beat the Oracle?

Try playing interactively and see how close you can get to the oracle's performance!

```
env = BernoulliBanditEnvironment(2)
n_steps = 10

interactive_agent = InteractiveAgent(len(env.reward_dists))
total_reward = 0
# Uncomment to be interactive
#total_reward = env.simulate(interactive_agent, n_steps=n_steps)
oracle_reward = env.simulate(OracleAgent(env), n_steps=10000, verbosity=0)/10000 * n_steps
print(f'\nThe probabilities were {[dist.probs.item() for dist in env.reward_dists]}')
print(f'\nYour reward was ${total_reward}, the oracle policy would have received an award close to ${oracle_reward:.2f}')
```

```
The probabilities were [0.7707958817481995, 0.2072622925043106]

Your reward was $0, the oracle policy would have received an award close to $7.76
```

Demo: Random Agent (The Baseline)

- No learning involved
- Provides a **lower bound** on performance

```python
class RandomAgent:
  def __init__(self, n_actions):
    self.n_actions = n_actions

  def select_action(self):
    return torch.randint(self.n_actions, (1,))[0]

  def update(self, action_index, reward):
    return self

  def __str__(self):
    return f'Random'

torch.manual_seed(0)
env = BernoulliBanditEnvironment(3)
random_agent = RandomAgent(len(env.reward_dists))
env.simulate(random_agent, n_steps=10)
```

```
Step=01, Action=2, Reward=0, Cum reward=0, Avg reward=0.00,
Step=02, Action=1, Reward=1, Cum reward=1, Avg reward=0.50,
Step=03, Action=0, Reward=1, Cum reward=2, Avg reward=0.67,
Step=04, Action=1, Reward=0, Cum reward=2, Avg reward=0.50,
Step=05, Action=2, Reward=0, Cum reward=2, Avg reward=0.40,
Step=06, Action=1, Reward=1, Cum reward=3, Avg reward=0.50,
Step=07, Action=1, Reward=0, Cum reward=3, Avg reward=0.43,
Step=08, Action=2, Reward=0, Cum reward=3, Avg reward=0.38,
Step=09, Action=2, Reward=0, Cum reward=3, Avg reward=0.33,
Step=10, Action=1, Reward=1, Cum reward=4, Avg reward=0.40,

tensor(4.)
```

Greedy Agent: Implementation Details

Note: Initialization of Q matters. Higher initial values → more early exploration.

```python
class GreedyAgent:
  def __init__(self, n_actions, init_value=0):
    self.init_value = init_value
    self.action_counts = torch.zeros(n_actions) #n_t
    self.action_value_func = init_value * torch.ones(n_action

  def select_action(self):
    return torch.argmax(self.action_value_func)

  def update(self, action_index, reward):
    action_count = self.action_counts[action_index]
    sum_rewards = action_count * self.action_value_func[actio
    new_avg_reward = (sum_rewards + reward) / (action_count +
    self.action_counts[action_index] += 1
    self.action_value_func[action_index] = new_avg_reward
    return self

  def __str__(self):
    return f'Greedy(init={self.init_value})'

torch.manual_seed(0)
env = BernoulliBanditEnvironment(3)
# Try init_value = 0, 1 or 100
greedy = GreedyAgent(len(env.reward_dists), init_value=1)
env.simulate(greedy, n_steps=20)
```

```
Step=01, Action=0, Reward=1, Cum reward=1, Avg reward=1.00,
Step=02, Action=0, Reward=1, Cum reward=2, Avg reward=1.00,
Step=03, Action=0, Reward=0, Cum reward=2, Avg reward=0.67,
Step=04, Action=1, Reward=1, Cum reward=3, Avg reward=0.75,
Step=05, Action=1, Reward=1, Cum reward=4, Avg reward=0.80,
Step=06, Action=1, Reward=0, Cum reward=4, Avg reward=0.67,
Step=07, Action=2, Reward=0, Cum reward=4, Avg reward=0.57,
Step=08, Action=0, Reward=0, Cum reward=4, Avg reward=0.50,
Step=09, Action=1, Reward=1, Cum reward=5, Avg reward=0.56,
Step=10, Action=1, Reward=0, Cum reward=5, Avg reward=0.50,
Step=11, Action=1, Reward=1, Cum reward=6, Avg reward=0.55,
Step=12, Action=1, Reward=1, Cum reward=7, Avg reward=0.58,
Step=13, Action=1, Reward=1, Cum reward=8, Avg reward=0.62,
Step=14, Action=1, Reward=0, Cum reward=8, Avg reward=0.57,
Step=15, Action=1, Reward=0, Cum reward=8, Avg reward=0.53,
Step=16, Action=1, Reward=0, Cum reward=8, Avg reward=0.50,
Step=17, Action=1, Reward=0, Cum reward=8, Avg reward=0.47,
Step=18, Action=0, Reward=1, Cum reward=9, Avg reward=0.50,
Step=19, Action=0, Reward=1, Cum reward=10, Avg reward=0.53,
Step=20, Action=0, Reward=1, Cum reward=11, Avg reward=0.55,
tensor(11.)
```

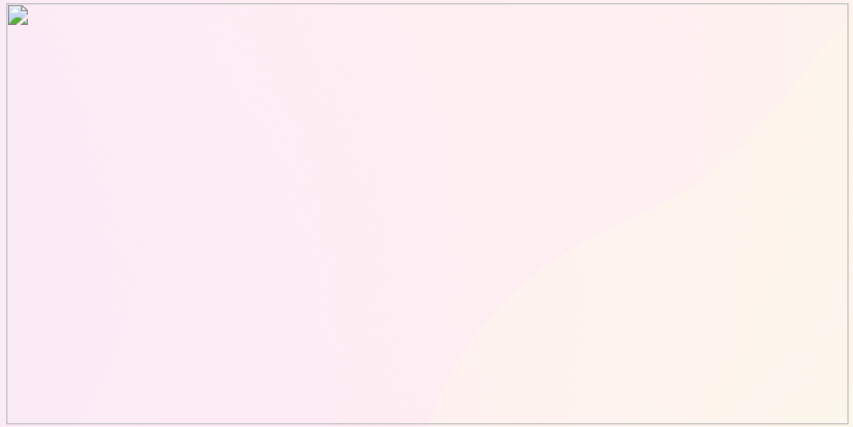
ε-Greedy Implementation

```python
class EpsilonGreedyAgent(GreedyAgent):
  def __init__(self, n_actions, epsilon):
    super().__init__(n_actions, init_value=0)
    self.epsilon = epsilon

  def select_action(self):
    if torch.rand((1,))[0] < self.epsilon:
      return torch.randint(len(self.action_value_func), (1,))
    else:
      return torch.argmax(self.action_value_func)

  def __str__(self):
    return f'$\epsilon$-Greedy($\epsilon={self.epsilon:.2f}$)'

# Try init_value = 0, 1 or 100
torch.manual_seed(0)
env = BernoulliBanditEnvironment(3)
epsilon_greedy = EpsilonGreedyAgent(len(env.reward_dists), ep
env.simulate(epsilon_greedy, n_steps=20)
```

```
Step=01, Action=0, Reward=1, Cum reward=1, Avg reward=1.00,
Step=02, Action=0, Reward=1, Cum reward=2, Avg reward=1.00,
Step=03, Action=0, Reward=1, Cum reward=3, Avg reward=1.00,
Step=04, Action=0, Reward=1, Cum reward=4, Avg reward=1.00,
Step=05, Action=0, Reward=1, Cum reward=5, Avg reward=1.00,
Step=06, Action=1, Reward=0, Cum reward=5, Avg reward=0.83,
Step=07, Action=0, Reward=0, Cum reward=5, Avg reward=0.71,
Step=08, Action=0, Reward=1, Cum reward=6, Avg reward=0.75,
Step=09, Action=0, Reward=1, Cum reward=7, Avg reward=0.78,
Step=10, Action=2, Reward=0, Cum reward=7, Avg reward=0.70,
Step=11, Action=0, Reward=1, Cum reward=8, Avg reward=0.73,
Step=12, Action=0, Reward=0, Cum reward=8, Avg reward=0.67,
Step=13, Action=0, Reward=0, Cum reward=8, Avg reward=0.62,
Step=14, Action=0, Reward=0, Cum reward=8, Avg reward=0.57,
Step=15, Action=0, Reward=1, Cum reward=9, Avg reward=0.60,
Step=16, Action=2, Reward=0, Cum reward=9, Avg reward=0.56,
Step=17, Action=0, Reward=0, Cum reward=9, Avg reward=0.53,
Step=18, Action=0, Reward=1, Cum reward=10, Avg reward=0.56,
Step=19, Action=0, Reward=1, Cum reward=11, Avg reward=0.58,
Step=20, Action=0, Reward=1, Cum reward=12, Avg reward=0.60,

tensor(12.)
```

Comparing All Algorithms

```python
# Compare algorithms
n_actions = 100
n_steps = 1000
agents = [
  RandomAgent(n_actions),
  GreedyAgent(n_actions, init_value=0),
  GreedyAgent(n_actions, init_value=1),
  EpsilonGreedyAgent(n_actions, epsilon=0.0),
  EpsilonGreedyAgent(n_actions, epsilon=0.01),
  EpsilonGreedyAgent(n_actions, epsilon=0.1),
]
label_list = []
mean_reward_tensor = []
for seed in range(30):
  mean_reward_list = []
  torch.manual_seed(seed)
  env = BernoulliBanditEnvironment(n_actions)
  #print([dist.probs.item() for dist in env.reward_dists])
  for agent in agents + [OracleAgent(env)]:
    torch.manual_seed(seed*1000)
    mean_reward = []
    cum_reward = 0
    for step in range(n_steps):
      _, reward = env.simulate_step(agent)
      cum_reward += reward
      mean_reward.append(cum_reward / (step + 1))
    mean_reward_list.append(mean_reward)
```



Key Takeaways

What we learned:

- **Random agent**: Worst performance, pure exploration baseline
- **Greedy (init=0)**: Can get stuck on suboptimal actions
- **Greedy (init=1)**: Optimistic initialization helps early exploration
- **ε-Greedy**: Best long-term performance with balanced exploration

The exploration-exploitation dilemma: - Too much exploration → waste time on bad actions - Too much exploitation → miss better alternatives

Non-stationary / dynamic bandits relax the assumption that the environment is only in one state

- The distribution of rewards changes over time.
  - Though this doesn't necessarily mean that the actions affect the environment.
- The optimal q_* is dependent on time.
- In practice, the estimate Q_t can be updated using a gradient-like rule:

$$Q_{t+1}(A_t) \triangleq Q_t(A_t) + \alpha[R_t - Q_t(A_t)]$$

- This turns out to be a decaying weighted average (i.e., more weight on the most recent rewards):

$$Q_{t+1}(A_t) = (1-\alpha)^t Q_1(A_t) + \sum_{i=1}^{t} \alpha(1-\alpha)^{t-i} R_i$$

Contextual bandits relax the assumption that the agent can observe some clue about the environment state

- Suppose now that the environment changes (nonstationary) **AND** that the agent can observe some clue or **contextual information** about the environment.
- **Examples:**
  - Netflix images – The demographics or previous ratings of the user.
  - Best search result – The search query and user history.
- Now the best action depends on this **context**, or more generally some observation of the **environment state**, denoted S_t.
  - This is the "input" to the action-selection algorithm (like x_i for supervised learning)
- One remaining assumption is that the actions do **NOT** affect the **next state**.
  - Thus, there is still **no notion of planning** in contextual bandits.
  - This is the last remaining assumption to relax to get the full RL problem.

Summary

- Multi-armed bandits are a simplification of RL.
  - Single environment state.
  - Lack of knowledge / uncertainty is the key challenge.
- Bandit problems retain key unique aspects of RL including:
  - **Evaluative feedback** rather than instructive feedback.
  - **Explore-exploit tradeoff**.
- Bandit algorithms
  - Random, Greedy, ϵ-Greedy.
- Variants of bandit problems
  - **Nonstationary bandits** – Environment changes over time.
  - **Contextual bandits** – Agent observes clues/context about the environment state.
  - Both assume that actions do NOT affect future environment states.

Reference

- Based on the excellent RL book by Sutton and Barto:
  - http://incompleteideas.net/book/the-book-2nd.html